OPTIMAL POLICIES FOR QUEUING SYSTEMS WITH PERIODIC REVIEW

By

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To

Roger Eric

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The general problem of finding optimal policies for operating queuing systems is investigated in order to determine when to open or close servers so as to minimize cost. The queue size is observed at equally spaced intervals of time and the structure of the optimal policy is determined.

The main restriction made, at first, is to limit the model to a single server system. Justification for a form of an optimal policy is given. Then the proofs that this policy is in fact optimal for the finite horizon, infinite horizon, and averaging cost criterion are given. Also, conditions for determining that the server should never be closed with customers in queue are presented.

Two computational schemes for determining the optimal policy are presented. The first, utilizing system parameters, minimizes average cost by computing the expected waiting time. The other solves the infinite horizon discounted cost criterion by direct use of Howard's Markov programming algorithm.

The problem is then generalized to allow for a fixed number of servers. Most of the results from the single server follow immediately. The main difficulty is, therefore, to establish and justify the desired rule so that it is analogous with the single server and is still as general as possible.

The primary mathematical concept used in solving the problem is dynamic programming. This technique yields the solution to the finite and infinite discounted cost criteria in both models presented. Some of the results from replacement theory help in solving the averaging case.

Finally, the conclusion shows how this work is an extension of the related work in the field, and points out several extensions which naturally follow. The fact that the form of the optimal policy is both intuitively appealing and easy to implement makes the results useful in many queuing and related systems, such as inventory and replacement models.

CHAPTER I

INTRODUCTION AND PROBLEM STATEMENT

Introduction

An important problem to consider is the economic behavior of a stochastic service system. Most research in queuing analysis has dealt with the underlying probabilistic structure rather than identifying optimal policies for the operation of the system. Within the operation of a service station there is a fraction of time in which the server is idle. This idle fraction can be eliminated in one of two ways. First, the server can carry out auxiliary duties during this idle fraction or, as an alternative, there can be a partial or complete elimination of the station. We are concerned with the latter, i.e., when to open or close our station in order that some objective is optimized. The solution to this problem is by no means immediately obvious, especially when there is a statistical input of customers. In this paper we will discuss a reasonable form for an optimal solution, prove under what conditions this solution exists and discuss extensions including computational refinements for this type of solution.

Previous Results

The justification for this work stems from the similarity

between queuing and inventory models. Much work has been done using explicit cost functions on inventory models[1,7] in particular getting optimal policies of a particular structure, such as the s-S policy. Because of this similarity in structure it seemed reasonable to attempt to fit explicit cost functions to the model with the hope of finding optimal policies similar in structure to the s-S policy.

The mathematical background for this dissertation comes first from dynamic programming[2,13] and then from many papers on Markovian Decision Processes.[9,10] Many of these works are discussed in a survey paper by the author.[11] In a paper on replacement policies which is an application of sequential decision process, Derman[5] defines a class of rules as control limit rules. The form of this rule was helpful in obtaining optimal rules for the queuing model.

The earliest paper on optimization of queuing systems was by Edie.[6] He uses optimization techniques in determining the best level of service based on traffic volume at toll booths. Then the number of toll booths required at any time of day can be established. Edie's results provided a minimum number of toll collectors providing acceptable service with relief for the toll collectors.

A paper by Yadin and Naor[17] was an important inspiration for this work. They dealt with the partial elimination of a single service station under a (O,R) doctrine. This is where the server when closed remains closed until R customers are present in the system and remains open until the system

is empty. They give properties of this system under this rule and also give computational aspects leading to a closed form solution for the value of R.

Most recently Heyman[8] and Sobel[15] have recognized the importance of explicit cost functions for stochastic service systems. Heyman considers a single channel queue with Poisson arrivals and arbitrary service time distribution. Upon completion of arrival and service epochs he shows that for infinite horizon models there exists an optimal stationary policy of particular form. He also finds an explicit cost function and thereby is able to find an optimal policy as a function of the parameters of the system. Heyman next considers a two-channel model, one of which is always open, with the added assumption that the service time distribution is exponential. He again shows what form the optimal policy must take and uses Jewell's Markov Renewal Programming technique to find the policy.

Sobel generalizes the problem by making the arrival distribution arbitrary. Also, where Heyman uses a linear holding cost, Sobel uses a very general holding cost. He then shows existence for the same type of optimal policy as does Heyman.

General Problem Statement

Consider "customers" arriving at a service facility from an infinite population according to some statistically steady input stream with distribution function A(t). Also, assume

there are at most s servers, each with distribution of service time $B_k(t)$, $k = 1, 2, \ldots, s$. The system is observed at discrete times $(t_1, t_2 \ldots)$. We are interested in first showing the existence and then finding a policy of form (A) so as to minimize some cost structure, where form (A) is:

(A) for each k ($k = 1, 2, \ldots, s$) identify a unique ordered pair $[a^*(t), b^*(t)]$ such that if i, the number in the system, falls between $a^*(t)$ and $b^*(t)$ a specific action will be determined. If i does not fall between these values then no action will be taken. By an action we mean how many servers should be opened or closed . The form of the optimal policy for each specific problem will be of this structure and will also inherently make it clear how many servers should be opened or closed.

This differs from the aforementioned works because of the following refinements:

(1) Assume exponential inter-arrival and service distributions.

(2) Allow for a fixed and finite number of servers.

(3) Show the existence of an optimal policy for finite horizon, infinite horizon and averaging cost criteria.

(4) Make the decision points at the end of time periods of equal length. This is a major difference between this work and the others. Observing at the end of equal time intervals causes many difficulties that are not encountered when observation points are the completion of arrival or service epochs. These will be pointed our later in the text.

CHAPTER II

SINGLE SERVER MODEL

Let us first consider a modification of this problem which is far more tractable.

A Specific Problem

As a specific problem let us make the following changes from the general problem formulation.

(I) Assume there is just one server. This is done solely to simplify the problem,and hopefully will give insight to the multiserver case.

(II) Assume the system has no bounds on capacity. But also assume there is only a finite number of states for which decisions will be made, i.e. when observed with more than M customers, for some finite M, the decision will always be to open if we are closed. This is not a major restriction and the reason for it will be pointed out later in the text.

(III) Assume stationary solutions. Blackwell[3] assures that both a* and b* will not be functions of time, in an infinite horizon model, because of the finiteness of the action space (here, only two possible decisions).

(IV) Assume both the times between arrivals and services are distributed exponentially. Some comments are needed to

justify this assumption. A desirable characteristic of a real-life service system is that the monitoring or checking times are not too frequent. In fact, we would like to observe our system at equal time intervals (once a minute, hour, day, etc.). To ensure the problem is solvable the sequence of states must form a Markov Chain (number of customers at our service station must be independent of the past history of the process); that is, there must be stationary transition probabilities. This implies that the number of arrivals and services must only depend on the length of the time period. In order not to increase the dimensionality of the state space, A(t) and B(t) are restricted to be exponential distribution functions. If, for example, B(t) has a general probability distribution, to ensure the Markovian property it would be necessary to observe the system at the completion of each service when it was open. We have chosen to be more general in our observation points than in the arrival and service distributions. However, much of the analysis and many of the results can be used for the other, more general, case.

The specific problem can now be stated. Customers arrive before a single server according to the Poisson probability law. The time between services is exponential. The system is observed at equal time intervals t = 1, 2, ... We are interested in first showing the existence and then finding a policy of form (A') so as to minimize some cost function; where (A') is:

(A') identify a unique pair (j*, i*) with the interpretation that if and only if the server is open when

observed and there are at most j^* customers present will the system be closed and if and only if the server is closed and there are at least i^* customers present when observed will the server be opened.

Note that this policy could be expressed precisely in the form of (A) from Chapter I. If for $k = 0$, no servers open, we choose (a^*, b^*) to be(i^*, ∞) then the action will be to open the server if the number in the system is between a^* and b^*, and no action would be taken otherwise. Similarly, if $k = 1$ and (a^*, b^*) is chosen as $(-\infty, j^*)$ then the server will only be closed when the number of customers is between a^* and b^*.

This differs with Yadin and Naor in that they <u>assumed</u> a (O,R) doctrine and computationally found the optimal value of R.

<u>Characteristics</u>

The properties of this model can now be given as follows:

(a) The Arrival Pattern -- Customers arrive so that the time between successive arrivals is distributed exponentially.

(b) The Service Mechanism -- Customers are served in order of their arrival so that the time between successive services is distributed exponentially.

(c) Cost Structure --

(i) A - Constant cost of opening a closed server

(ii) B - Constant cost of closing an open server

(iii) C - Cost of operating the server for one time period

(iv) K(i) - Holding cost, i.e., cost incurred when i customers are observed in the system.

These costs may or may not be discounted with discount factor α.

The decision process is as follows. The system is observed at times $t = 1, 2, \ldots$ and the state of the system X_t is noted. $X_t = i$ implies there are i customers in the system at time t. The decision is then made to either leave the server in its present state (opened or closed) or to open a closed server or to close an open server, or if $X_t \geq M$ and the server is closed it must be opened.

The sequence of states $\{X_i\}$ forms a Markov Chain and changes of states occur according to the following probabilistic structure,

$$p_{ij} = \Pr \{X_{t+1} = j | X_t = i \text{ and the server is closed}\}$$

$$q_{ij} = \Pr \{X_{t+1} = j | X_t = i \text{ and the server is open}\}$$

Existence of a Control Limit Rule

Consider a single channel system with the aforementioned characteristics. We would like to establish the existence of a control limit rule. This is, if the server is closed show that there exists a unique number i*, such that if the number of customers observed in the system is no less than i* then open the server. Further, if the number observed is less than i* then keep the server closed. This rule has intuitive appeal, saying, that if the server is not open with, say, 7 customers in queue then certainly it will not be open with, say, 5.

It is desirable to show the existence of this number for an averaging objective function. In doing so we obtain similar results for both the discounted finite and infinite horizon models.

Dynamic Programming Formulation

Consider the solution to the finite horizon model using dynamic programming. Define,

$\phi_n^k(i,\alpha)$ = minimum expected cost with n time periods remaining when i customers are observed in the queue using discount factor α, and k = 0, 1 depending on whether the server is closed or open respectively.

Then the recursive equations are

$$\phi_n^0(i,\alpha) = \min \left\{ \begin{array}{l} K(i) + \alpha \sum_{j=0}^{\infty} p_{ij}\, \phi_{n-1}^0(j,\alpha) \\ \\ K(i) + A + C + \alpha \sum_{j=0}^{\infty} q_{ij}\, \phi_{n-1}^1(j,\alpha) \end{array} \right\}$$

where the alternatives are to keep the server closed or to open it.

And $$\phi_n^1(i,\alpha) = \min \left\{ \begin{array}{l} K(i) + B + \alpha \sum_{j=0}^{\infty} p_{ij}\, \phi_{n-1}^0(j,\alpha) \\ \\ K(i) + C + \alpha \sum_{j=0}^{\infty} q_{ij}\, \phi_{n-1}^1(j,\alpha) \end{array} \right\}$$

with alternatives closing the server or keeping it open, where it is assumed that

$$\phi_n^0(m,\alpha) = K(m) + A + C + \sum_{j=0}^{\infty} q_{mj}\, \phi_{m-1}^1(j,\alpha) \text{ for } m \geq M$$

Define,

$$\phi_0^k(i,\alpha) = K(i) \quad , \; k = 0, 1$$

i.e., the terminal cost of our process is the holding cost for the number of customers observed. In addition, assume

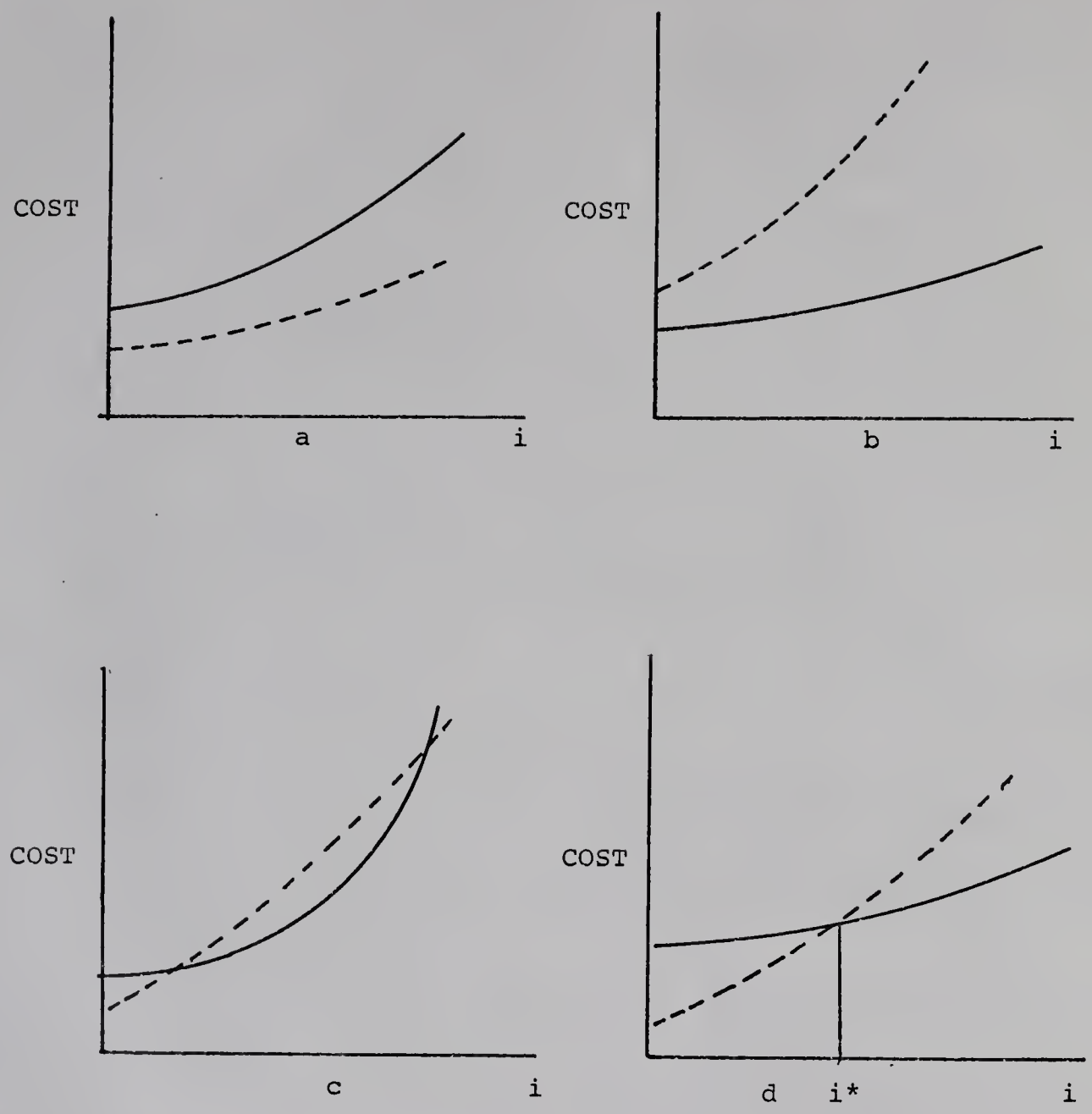
(i) that K(i) is a nondecreasing function of i, a reasonable assumption and one often made in inventory or replacement analysis.

and (ii) that for any nondecreasing function h(j), $\sum_{j=0}^{\infty} p_{ij} h(j)$ and $\sum_{j=0}^{\infty} q_{ij} h(j)$ are nondecreasing functions of i. This says, given some subset of states K = {k, k+1, ...} , the Chance of entering K is nondecreasing in i.

Consider the server closed. For n = 1 we have

$$\phi_1^0(i,\alpha) = \min \left\{ \begin{array}{l} K(i) + \alpha \sum_{j=0}^{\infty} p_{ij} \; K(j) \\ \\ K(i) + A + C + \alpha \sum_{j=0}^{\infty} q_{ij} \; K(j) \end{array} \right\} \quad (1)$$

From assumptions (i) and (ii) we know both of these alternatives and also, $\phi_1^0(i,\alpha)$ are nondecreasing. If at i = 0, the second alternative is chosen then we must ensure it must be chosen for all i (always open server). If at i = 0 the first alternative is chosen then these two nondecreasing curves may cross in at most one place, the i* being sought. If they don't cross at all before M then the server will never be opened until at least M customers are present. The possibilities are illustrated in Figure 1. To ensure case c cannot occur an additional assumption must be made.



a - Never open server

b - Never close server

c - Undesirable case

d - Unique opening point

- - - - cost for closed server

_____ cost for open server

Figure 1: Associated costs for two alternatives

We should comment here how Heyman's model, that of checking after each arrival, fails to encounter this difficulty. Let us say that case c denotes the graphs of the curves in Equation 1. Let i_1 be the first crossing point and $i_2 (>i_1)$ the second, that is keep the server closed for $i < i_1$ and $i \geq i_2$ and open otherwise. If the system is observed at $i < i_1$ and then at each arrival thereafter, as Heyman does, as soon as i_1 is reached the server will be opened and the state $i \geq i_2$ for a closed server will never be reached. Checking at discrete time points, however, may result for some t in

$$X_t = i \quad \text{for} \quad i < i_1$$

and

$$X_{t+1} = j \quad \text{for} \quad j > i_2$$

resulting in keeping the server closed at both points, and hence his rule of always opening after i_1 may not hold.

Additional Restriction

As so often happens in this type of problem when a barrier is reached, an additional condition must be imposed. By analogy to the definition of convexity we impose the following condition:

$$\theta K(j) + (1-\theta)\, K(i) \geq K(\theta j + (1-\theta) i) \text{ for}$$

$$\theta \varepsilon [0,1] \text{ and the argument for } K(.) \text{ integral} \qquad (2)$$

For obvious reasons we call (2) the convex property. It would be possible to achieve the same results by defining the envelope of the function as convex. However, there is very little additional computation in analyzing the discrete function.

Uniqueness Proof for n = 1

Recalling that

$$\phi_1^0(i,\alpha) = \min \left\{ \begin{array}{l} K(i) + \alpha \sum_{j=0}^{\infty} p_{ij} K(j) \\ \\ K(i) + A + C + \alpha \sum_{j=0}^{\infty} q_{ij} K(j) \end{array} \right\}$$

it must be determined when

$$K(i) + \alpha \sum_{j=0}^{\infty} p_{ij} K(j) \ (\lesseqgtr) \ K(i) + A + C + \alpha \sum_{j=0}^{\infty} q_{ij} K(j)$$

or

$$\sum_{j=0}^{\infty} K(j) \ [p_{ij} - q_{ij}] \ (\lesseqgtr) \ \frac{C + A}{\alpha} \triangleq A$$

This will be done by showing that

$$\sum_{j=0}^{\infty} K(j) \ [p_{ij} - q_{ij}]$$

is a nondecreasing function of i and hence can cross the constant line A at most once.

Let x and y be the number of arrivals and services, respectively, in a given time period.

Then

$$\sum_{j=0}^{\infty} p_{ij} \ K(j) = \sum_{x} \Pr \text{ (number of arrivals equals } x) \cdot K(i+x)$$

$$= E_x \ K(i + x) = E_x \ E_y \ K(i + x)$$

and also,

$$\sum_{j=0}^{\infty} q_{ij} \ K(j) = \sum_{x} \sum_{y} \Pr(x \text{ arrivals}) \cdot \Pr \text{ (y services)} \cdot K(i+x-y)$$

$$= E_x E_y \; K(i+x-y)$$

where

$$K(n) = 0 \text{ for } n < 0.$$

Since K is nondecreasing

$$D_i \triangleq E_x E_y \; [K(i+x) - K(i+x-y)] \geq 0^{[1]}$$

or,

$$\sum_{j=0}^{\infty} K(j) \; [p_{ij} - q_{ij}] \geq 0$$

Now, allowing $j = i + h$, (2) can be rewritten as

$$\theta K(i + h) + (1 - \theta) \; K(i) \geq K(\theta i + \theta h + i - \theta i)$$

$$= K(i + \theta h)$$

or

$$\theta K(i + h) - \theta K(i) \geq K(i + \theta h) - K(i)$$

---

[1] To say this we must ensure that both $\Sigma \, p_{ij} K(j)$ and $\Sigma \, q_{ij} K(j)$ are finite. To show convergence employ the ratio test as follows:

$$\sum_{j=0}^{\infty} p_{ij} K(j) = \sum_{j=i}^{\infty} \frac{\lambda^{(j-i)} e^{-\lambda}}{(j-i)!} \; K(j)$$

so that

$$p_{i,j+1} K(j+1) = \frac{\lambda^{j-i+1} e^{-\lambda}}{(j-i+1)!} \; k(j+1) \text{ and}$$

$$p_{i,j} K(j) = \frac{\lambda^{j-i+1} e^{-\lambda}}{(j-i)'} \text{ and } K(j) \text{ and the limit as } j \to \infty \text{ of}$$

$$\frac{p_{i,j+1} K(j+1)}{p_{i,j} \; K(j)} = \frac{\lambda}{(j-i+1)} \; \frac{K(j+1)}{K(j)} \text{ must be } < 1.$$

This holds for a wide class of functions such as exponential and polynomials and will be assumed true. Certainly, then, $\Sigma \, q_{ij} K(j)$ will also converge as it is less than or equal to $\Sigma \, p_{ij} K(j)$.

or as an alternate definition

$$K(i + h) - K(i) \geq \frac{1}{\theta} [K(i + \theta h) - K(i)] \qquad (2')$$

which must hold for all $0 \leq \theta \leq 1$ and θh integer. In particular it must hold for $\theta h = 1$, i.e.,

$$K(i + h) - K(i) \geq h[K(i + 1) - K(i)]$$

for all integral h. As a special case when $h = 2$

$$K(i + 2) - K(i) \geq 2 [K(i + 1) - K(i)]$$

or

$$K(i + 2) - 2 K(i + 1) + K(i) \geq 0 \qquad (3)$$

Denote by $\Delta^{(j)}K$ as the jth difference of $K(i)$. Then

$$\Delta^{(1)}K = K(i + 1) - K(i)$$

$$\Delta^{(2)}K = \Delta(\Delta^{(1)}K) = K(i + 2) - 2 K(i + 1) + K(i)$$

Notice, however, that this second equation is just equation (3). Hence, we have proven the following lemma.

Lemma 1: The second difference of functions with the convex property is non-negative.

Therefore, we can conclude that

$$\Delta^{(1)}K = K(i + 1) - K(i) \text{ is nondecreasing.} \qquad (4)$$

Now consider

$$\Delta^{(1)} [K(i + x) - K(i + x - y)] =$$

$$K(i + x + 1) - K(i + x) - K(i + x - y + 1) + K(i + x - y)$$

But this is non-negative since

$$K(i + x + 1) - K(i + x) \geq K(i + x + 1 - y) - K(i + x - y)$$

from the property in (4). Hence,

$$K(i + x) - K(i + x - y)$$

is nondecreasing in i for all x and y. Then surely,

$$D_i = E_x E_y [K(i + x) - K(i + x - y)]$$

must be nondecreasing in i and hence crosses A at most once. This proves the following important theorem.

Theorem 1: For $n=1, \phi_n^0(i, \alpha)$ has a solution given by a control limit rule, i.e., there exists a unique number such that if and only if with one period remaining there are more than this number in the system, will a closed server be opened.

For the case when the server is open with one period remaining,

$$\phi_1^1(i, \alpha) = \min \left\{ \begin{array}{l} K(i) + B + \alpha \sum_{j=0}^{\infty} p_{ij} K(j) \\ \\ K(i) + C + \alpha \sum_{j=0}^{\infty} q_{ij} K(j) \end{array} \right\}$$

This leads to observing when

$$K(i) + B + \alpha \Sigma p_{ij} K(j) \ (\lesseqgtr) \ K(i) + C + \alpha \Sigma q_{ij} K(j)$$

or,

$$\sum_{j=0}^{\infty} [p_{ij} - q_{ij}] K(j) \ (\lesseqgtr) \ \frac{C-B}{\alpha} = \beta$$

The preceding theorem then also applies here, hence guaranteeing a unique point at which the server would be closed.

<u>Proof for arbitrary n</u>

For n>1 it must be shown that (j*, i*) policies are optimal for,

$$\phi_n^0(i, \alpha) = \min \left\{ \begin{array}{l} K(i) + \alpha \sum_{j=0}^{\infty} p_{ij} \phi_{n-1}^0 (j, \alpha) \\ \\ K(i) + A + C + \alpha \sum_{j=0}^{\infty} q_{ij} \phi_{n-1}^1 (j, \alpha) \end{array} \right\}$$

and

$$\phi_n^1(i, \alpha) = \min \left\{ \begin{array}{l} K(i) + B + \alpha \Sigma p_{ij} \phi_{n-1}^0 (j, \alpha) \\ \\ K(i) + C + \alpha \Sigma p_{ij} \phi_{n-1}^1 (j, \alpha) \end{array} \right\}$$

Before this is shown, let us prove the following lemma.

Lemma 2: For $n=1$, $i^* \geq j^*$

Proof: For $n=1$, i^* must satisfy

$$K(i^*) + \alpha \Sigma p_{i^*j} K(j) \geq K(i^*) + A + C + \alpha \Sigma q_{i^*j} K(j) \text{ or,}$$

$$\Sigma [p_{i^*j} - q_{i^*j}] K(j) \geq \frac{A+C}{\alpha}$$

Similarly j^* must satisfy, $K(j^*) + B + \alpha \Sigma p_{j^*j} K(j)$

$\leq K(j^*) + C + \Sigma q_{j^*j} K(j)$ or,

$$\Sigma [p_{j^*j} - q_{j^*j}] K(j) \leq \frac{C-B}{\alpha}$$

Since $A + C > C - B$ and $\Sigma [p_{ij} - q_{ij}] K(j)$ is nondecreasing, $i^* \geq j^*$ must hold.

Now let us establish the existence of the values (j^*, i^*) for $n=2$. We have that

$$\phi_2^0(i, \alpha) = \min \left\{ \begin{array}{l} K(i) + \alpha \Sigma p_{ij} \phi_1^0 (j, \alpha) \\ \\ K(i) + A + C + \alpha \Sigma q_{ij} \phi_1^1 (j, \alpha) \end{array} \right\} \quad (5)$$

and

$$\phi_2^1(i,\alpha) = \min \left\{ \begin{array}{l} K(i) + \alpha \Sigma p_{ij} \ \phi_1^0(j,\alpha) + B \\ K(i) + C + \alpha \Sigma q_{ij} \ \phi_1^1(j,\alpha) \end{array} \right\} \qquad (6)$$

Just as before we can write

$$\Sigma p_{ij} \ \phi_1^0(j,\alpha) = E_x \ E_y \ \phi_1^0(i+x,\alpha)$$

and

$$\Sigma q_{ij} \ \phi_1^1(j,\alpha) = E_x \ E_y \ \phi_1^1(i+x-y,\alpha),$$

and, as before, if it can be shown that $\phi_1^0(i+x,\alpha) - \phi_1^1(i+x-y,\alpha)$ is nondecreasing, then so is

$$\Sigma \ [p_{ij} \ \phi_1^0(j,\alpha) - q_{ij} \ \phi_1^1(j,\alpha)]$$

which implies that the alternatives in (5) and (6) can cross at most once and the result will be established.

$$\phi_1^0(i+x,\alpha) = \min \left\{ \begin{array}{ll} K(i+x) + \Sigma \alpha \ p_{i+x,j} \ K(j) & \text{(I)} \\ K(i+x) + A + C + \alpha \Sigma q_{i+x,j} \ K(j) & \text{(II)} \end{array} \right\}$$

$$- \min \left\{ \begin{array}{ll} K(i+x-y) + B + \alpha \Sigma p_{i+x-y,j} \ K(j) & \text{(III)} \\ K(i+x-y) + C + \alpha \Sigma q_{i+x-y,j} \ K(j) & \text{(IV)} \end{array} \right\} = \phi_1^1(i+x-y,\alpha)$$

where the (I), (II), (III) and (IV) notationally correspond to that particular alternative.

It is necessary to consider only the possibilities when the minima are (I) and (III), (I) and (IV), and (II) and (IV), i.e. (II) and (III) cannot concurrently be minima by lemma 2. Now, let us formally prove the following.

Theorem 2: $\phi_1^0(i+x,\alpha) - \phi_1^1(i+x-y,\alpha)$ is nondecreasing.

Proof: When (I) and (III) are minima $\phi_1^0(i+x,\alpha) - \phi_1^1(i+x-y,\alpha) =$

$$K(i+x) - K(i+x-y) + \alpha\,\Sigma[p_{i+x,j} - p_{i+x-y,j}]\,K(j) - B$$

which is nondecreasing, the first two terms from assumption(i) and the second two since $K(i+1) - K(i)$ is nondecreasing.

When (I) and (IV) are minima,

$$\phi_1^0(i+x,\alpha) - \phi_1^1(i+x-y,\alpha) = K(i+x) - K(i+x-y)$$

$$+ \alpha\,\Sigma\,[p_{i+x,j} - q_{i+x-y,j}]\,K(j) - C$$

which is nondecreasing. This can be shown be replacing the bracket by

$$[p_{i+x,j} - p_{i+x-y,j} + p_{i+x-y,j} - q_{i+x-y,j}]\,K(j)$$

which is the sum of two nondecreasing functions.

Finally when (II) and (IV) are minima

$$\phi_1^0(i+x) - \phi_1^1(i+x-y,j) = K(i+x) - K(i+x-y)$$

$$+ \alpha\,\Sigma\,[q_{i+x,j} - q_{i+x-y,j}]\,K(j) + A$$

which is nondecreasing as was the first case considered. This completes the proof.

This, then, leads to the following:

Corollary 1: There exist numbers (j^*, i^*) for $n=2$ such that $\phi_2^0(i,\alpha)$ and $\phi_2^1(i,\alpha)$ have optimal solutions of the required form.

Now, let us introduce the hypothesis that,

for $n=3, 4, \ldots, k, \phi_n^0(i,\alpha)$ and $\phi_n^1(i,\alpha)$ have control limit

rules and, also,

$$\Sigma\,[p_{ij}\,\phi^0_{n-1}(j,\alpha) - q_{ij}\,\phi^1_{n-1}(j,\alpha)]$$

is nondecreasing.

It becomes necessary to show next that

$$\phi^0_{k+1}(i,\alpha) = \min \left\{ \begin{array}{l} K(i) + \alpha\,\Sigma\,p_{ij}\,\phi^0_k(j,\alpha) \\ \\ K(i) + \alpha\,\Sigma\,q_{ij}\,\phi^0_k(j,\alpha) + A + C \end{array} \right\}$$

and

$$\phi^1_{k+1}(i,\alpha) = \min \left\{ \begin{array}{l} K(i) + B + \alpha\,\Sigma\,p_{ij}\,\phi^0_k(j,\alpha) \\ K(i) + C + \alpha\,\Sigma\,q_{ij}\,\phi^1_k(j,\alpha) \end{array} \right\}$$

have the required property. This is done in precisely the same manner as before, i.e., it is shown that

$$\Sigma\,[p_{ij}\,\phi^0_k(j,\alpha) - q_{ij}\,\phi^1_k(j,\alpha)]$$

is nondecreasing by looking at

$$\phi^0_k(i+x,\alpha) - \phi^1_k(i+x-y,\alpha)$$

Again, we shall first show

Lemma 3: For each n, $i^* \geq j^*$.

Proof: At i^*, the following must be true,

$$K(i^*) + \alpha\,\Sigma\,p_{i^*j}\phi^0_{n-1}(j,\alpha) \geq K(i^*) + A + C + \alpha\,\Sigma\,q_{i^*j}\phi^1_{n-1}(j,\alpha)$$

or

$$\Sigma\,[p_{i^*j}\phi^0_{n-1}(j,\alpha) - q_{i^*j}\phi^1_{n-1}(j,\alpha)] \geq \frac{A+C}{\alpha}$$

At j^* the following must hold,

$$K(j^*) + B + \alpha\,\Sigma\,p_{j^*j}\phi^0_{n-1}(j,\alpha) \leq K(j^*) + C + \alpha\,\Sigma\,p_{j^*j}\phi^1_{n-1}(j,\alpha)$$

or

$$\Sigma\,[p_{j^*j}\ \phi^0_{n-1}(j,\alpha) - q_{j^*j}\ \phi^1_{n-1}(j,\alpha)] \le \frac{C-B}{\alpha}$$

Since $A + C > C - B$ and the left sides are nondecreasing by the induction assumption, it naturally follows that $i^* \ge j^*$.

Theorem 3:

$$\phi^0_k(i+x,\alpha) = \min \begin{Bmatrix} K(i+x) + \alpha\,\Sigma\, p_{i+x,j}\ \phi_{k-1}(j,\alpha) & \text{(I)} \\ K(i+x) + \alpha\,\Sigma\, q_{i+x,j}\ \phi^1_{k-1}(j,\alpha) + A + C & \text{(II)} \end{Bmatrix}$$

$$-\min \begin{Bmatrix} K(i+x-y) + B + \alpha\,\Sigma\, p_{i+x-y}\ \phi^0_{k-1}(j,\alpha) & \text{(III)} \\ K(i+x-y) + C + \alpha\,\Sigma\, q_{i+x-y,j}\ \phi^1_{k-1}(j,\alpha) & \text{(IV)} \end{Bmatrix} = \phi^1_k(i+x-y,j)$$

(where the (I), (II), (III) and (IV) are unambiguous) is nondecreasing.

Proof: From lemma 3 we see once again that it is only necessary to consider when (I) and (III), (I) and (IV), and (II) and (IV) are minima.

When (I) and (III) are minima the difference is

$$K(i+x) - K(i+x-y) + \alpha\,\Sigma\,[p_{i+x,j} - p_{i+x-y,j}]\ \phi^0_{k-1}(j,\alpha) - B,$$

which is nondecreasing; the summation is nondecreasing which follows from the discussion on piecewise convex functions in Appendix I.

When the minima are (I) and (IV) we have

$$K(i+x) - K(i+x-y)$$

$$+ \alpha\,\Sigma\,[p_{i+x,j}\ \phi^0_{k-1}(j,\alpha) - q_{i+x-y,j}\ \phi^1_{k-1}(j,\alpha)] - C.$$

This is nondecreasing since

$$p_{i+x,j}\ \phi^0_{k-1}(j,\alpha) - q_{i+x-y,j}\ \phi^1_{k-1}(j,\alpha)$$

$$= [p_{i+x,j} - p_{i+x-y,j}]\ \phi^0_{k-1}(j,\alpha)$$

$$+ p_{i+x-y,j}\ \phi^0_{k-1}(j,\alpha) - q_{i+x-y,j}\ \phi^1_{k-1}\ (j,\alpha)$$

and the first two terms are nondecreasing by the first part of this proof and the second two by the induction hypothesis.

Finally, when (II) and (IV) are minima

$$K(i+x) - K(i+x-y) + \alpha\,\Sigma\,[q_{i+x,j} - q_{i+x-y,j}]\phi^1_{k-1}(j,\alpha) + A$$

is nondecreasing by again using the results of the discussion from piecewise convex functions. This completes the proof.

Hence, it is established that $\phi^0_{k+1}(j,\alpha)$ and $\phi^1_{k+1}(j,\alpha)$ have control limit optimal rules.

Thus, we have established,

Corollary 2: For any n, $\phi^0_n(i,\alpha)$ and $\phi^1_n(i,\alpha)$ have solutions which are control limit rules.

Extension to Other Objectives

The theory behind the next result is due largely to Bellman.[2]

Theorem 3: A control limit rule is optimal for the infinite horizon model.

Proof: Let us prove this theorem for $\phi^0_n(i,\alpha)$; the other follows similarly.

We know that

$$\phi_n^0(i,\alpha) = \min \left\{ \begin{array}{l} K(i) + \alpha \Sigma\, p_{ij}\, \phi_{n-1}^0\ (j,\alpha) \\ \\ K(i) + A + C + \alpha \Sigma\, q_{ij}\, \phi_{n-1}^1 (j,\alpha) \end{array} \right\}$$

Define,

$$T_1(\phi_{n-1}^0) = K(i) + \alpha \Sigma\, p_{ij}\, \phi_{n-1}^0 (j,\alpha)$$

and

$$T_2(\phi_{n-1}^0) = K(i) + A + C + \alpha \Sigma\, q_{ij}\, \phi_{n-1}^1 (j,\alpha)$$

Then,

$$\phi_n^o(i,\alpha) = \min_{j=1,2} \quad [T_j\ (\phi_{n-1}^0)]$$

Let $j(n)$ = index which gives the $\min_{j=1,2} [T_j(\phi_n)]$

which implies that

$$\phi_n^0(i,\alpha) = T_{j(n-1)}\ (\phi_{n-1}^0) \leq T_{j(n)}\ (\phi_{n-1}^0)$$

and

$$\phi_{n+1}^o(i,\alpha) = T_{j(n)}\ (\phi_n^0) \leq T_{j(n-1)} (\phi_n^0)$$

Subtracting,

$$|\phi_n^0(i,\alpha) - \phi_{n+1}^0(i,\alpha)| \leq \max\ [|T_{j(n-1)}(\phi_{n-1}^0) - T_{j(n-1)}(\phi_n^0)|,$$

$$|T_{j(n)}(\phi_n^0) - T_{j(n)}(\phi_{n-1}^0)|]$$

$$\leq \max_{j=1,2}[|T_j(\phi_{n-1}^0) - T_j(\phi_n^0)|]$$

$$\leq \max[|K(i) + \alpha \Sigma\, p_{ij} \phi_{n-1}^0 (j,\alpha)$$

$$- K(i) - \alpha \Sigma\, p_{ij} \phi_n^0 (j,\alpha)|, |K(i) +$$

$$A+C+\alpha \Sigma q_{ij}\phi^1_{n-1}(j,\alpha)-K(i)-A-C-\alpha \Sigma q_{ij}\phi^1_n(j,\alpha)| \ ]$$

$$\leq \max[|\alpha \Sigma p_{ij}\phi^0_{n-1}(j,\alpha)-\alpha \Sigma p_{ij}\phi^0_n(j,\alpha)|,$$

$$|\alpha \Sigma q_{ij}\phi^1_{n-1}(j,\alpha)-\alpha \Sigma q_{ij}\phi^1_n(j,\alpha)|]$$

$$\leq \alpha \Sigma p_{ij}|\phi^0_{n-1}(j,\alpha)-\phi^0_n(j,\alpha)|$$

Hence

$$\max_i|\phi^0_n(i,\alpha)-\phi^0_{n+1}(i,\alpha)|\leq\alpha \Sigma p_{ij} \max_i|\phi^0_{n-1}(i,\alpha)-\phi^0_n(i,\alpha)|$$

$$\leq\alpha \max_i|\phi^0_{n-1}(i,\alpha)-\phi^0_n(i,\alpha)|$$

Define

$$u_n = \max_i[|\phi^0_n(i,\alpha) - \phi^0_{n-1}(i,\alpha)|]$$

and hence

$$u_{n+1}\leq\alpha u_n\leq\alpha^2 u_{n-1}\leq \cdots\leq\alpha^n u_1$$

and therefore

$$\sum_{n=0}^{\infty} u_{n+1}\leq \sum_{n=0}^{\infty} \alpha^n u_1 = \frac{u_1}{1-\alpha}$$

and we conclude that

$$\sum_{n=0}^{\infty} [\phi^0_{n+1}(i,\alpha) - \phi^0_n(i,\alpha)]$$

converges uniformly and that $\phi^0_n(i,\alpha)$ converges to $\phi^0(i,\alpha)$ for all i, where $\phi^0(i,\alpha)$ satisfies,

$$\phi^0(i,\alpha) = \min \begin{bmatrix} K(i) + \alpha \Sigma p_{ij}\phi^0(j,\alpha) \\ K(i) + A + C + \alpha \Sigma q_{ij}\phi^1(j,\alpha) \end{bmatrix}$$

completing the proof.

Now the significance of restricting our decision set to be finite when the system is closed will become apparent. We shall assume that for any $\varepsilon > 0$, $\exists$ an $M(\varepsilon) \ni$: $Pr[X_t > M(\varepsilon)] < \varepsilon$ and, henceforth, consider opening only in a finite number of states. When the server is open, assuming that the arrival rate is less than the service rate is sufficient to guarantee that the queue size will remain finite; hence, only a finite number of decision points can occur.

Finally, we can prove the following:

Theorem 4: A control limit rule is optimal for the averaging cost criterion.

Proof: Let i_α be the optimal crossing point of $\phi^0(i,\alpha)$. Let $\{\alpha_s\}$, with $\lim_{s\to\infty} \alpha_s = 1$, be a sequence such that $i_{\alpha_s} = i^*$ for all s. Since there are at most a finite number of states where the system will be opened, such a sequence and i* must exist. Let R be any rule which is not control limit. Then certainly

$$\phi_R^0(i,\alpha_s) \geq \phi^0(i,\alpha_s).$$

Hence,

$$\lim_{s\to\infty} (1-\alpha_s) \quad \phi_R^0(i,\alpha_s) \geq \lim_{s\to\infty} (1-\alpha_s)\phi^0(i,\alpha_s)$$

and from Arrow, Karlin and Scarf,

$$\phi_R^0 = \lim_{s\to\infty} (1-\alpha_s) \ \phi_R^0(i,\alpha_s) \text{ where}$$

$$\phi_R^0 = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} g(X_t) \text{ and where}$$

$g(X_t)$ denotes the cost incurred when observed in state

X_t. Therefore, $\phi_R^0 \geq \phi^0$ where ϕ^0 is the optimal value under the control rule i* and hence i* is the optimal rule. The proof is identical for ϕ^1.

Closing When There is a Non-empty Queue

Many have conjectured that a server will only be closed if no customers are awaiting service. Others have said that the closing point is some function of the system parameters. Let us answer this question by showning under what conditions the following theorem is true.

Theorem 5: The server will never be closed when any customers are present in the system.

Let us assume that we have established a unique point, say i*, at which the server should be turned on or opened. Let us also assume, contrary to the theorem, that the optimal operating policy, Π, designates to close the server at some number j*, $0 < j^* < i^*$.

Since the process is in equilibrium, we know that with probability one, state j* (or less) will be reached once the server has been turned on. Hence, the process will alternate between states j* and i*. [2]

There exists, however, a number k, $0 < k < j^*$, such that the expected cost under policy Π is equivalent to the expected cost under policy Π_1, where,

Π_1: Open the server when there are i* - j* customers in

[2] In fact it will alternate between states less than or equal to j* and greater than or equal to i*.

line, close when the line is empty and keep k[3] customers on the side at all times.

Define now policy Π_2 as follows.

Π_2: Open the server when there are $i^* - j^*$ customers in line, close when line is empty, and after some finite time serve the k customers.

Now for each possible objective function consider the following three policies.

1) Total Expected Undiscounted Cost--Here the additional expected cost under Π_1, that of holding the k customers to the side becomes infinite, while the additional expected cost, that of serving the k customers, is certainly finite and hence Π_2 dominates Π_1 and, therefore, also Π.

2) Average Cost--The holding of the k customers under policy Π_1 occurs for each period and hence adds a positive quantity to the average period cost. However, the cost of serving the k customers occurs for only a finite number of periods and hence adds nothing to the average period cost. Hence, again Π_2 dominates both Π and Π_1.

3) Total Expected Discounted Cost--Here, both the holding cost of the k customers along with the cost of serving these k customers is finite and hence a comparison is necessary. For a discount factor α the cost of holding the k customers is at least

$$K(k) \sum_{n=0}^{\infty} \alpha^n = \frac{K(k)}{1-\alpha}$$

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[3] If k is not integer keep instead [k] customers off to the side, so if anything Π_1 dominates Π.

The cost of serving the k customers, which we assume takes ℓ periods, along with the holding cost of these customers, is bounded above by

$$K(k) \sum_{n=0}^{\ell-1} \alpha^n + C \sum_{n=0}^{\ell-1} \alpha^n$$

$$= [K(k) + C] \frac{1 - \alpha^\ell}{1 - \alpha}$$

To choose Π_2 over Π_1 we must require that

$$[K(k) + C] \frac{1 - \alpha^\ell}{1 - \alpha} < \frac{K(k)}{1 - \alpha}$$

$$\alpha^\ell > \frac{C}{K(k)+C}$$

Hence, if the discount factor is large (a reasonable assumption), Π_2 dominates Π as well as Π_1.

4) Finite Horizon-In this case it is possible to close the server with customers present. This event occurs if the holding cost is small compared to the operating cost.

Again, the difference between analyzing my system and that of analyzing the system after each arrival and departure is that k in the preceding argument is just j*.

Concluding Remarks

Finally we can say for the single server queue that an optimal stationary policy always has one of the following forms:

a) Always keep the server closed until M is reached. This case arises when discounting is used and the holding cost is small compared to the operating cost.

b) Never close the server. - This may occur when the server set-up and shut-down costs are large compared to the operating cost.

c) There exists a pair (j^*, i^*) with the interpretation, open the server only if there are at least i^* customers present; close the server when there are no more than j^* customers present. The important special case of $j^* = 0$ is discussed in Theorem 5 and that which followed.

CHAPTER III

COMPUTATIONS FOR SINGLE SERVER

Once existence of optimal solutions has been established it becomes necessary to find an i* which might make it worthwhile to "dismantle" the service station. The closing of the service station under a (0, i*) doctrine will cause increased holding, and set-up and shut-down costs. The savings from the partial elimination of the station, during an idle period, might not outweigh these increased costs. Hence, an optimal i* must be found and the cost therein achieved must be compared to that of a simple queue, i.e., never close the server. A simple computational scheme is developed to make this comparison.

The second part of this chapter makes use of Howard's approach in analyzing the system from a computational viewpoint. Description of the algorithm, calculation of probabilities,along with some interesting results are presented.

Averaging Computational Scheme

Calculation of Average Queue Length

Let us consider an averaging cost criterion, i.e., finding the optimal policy for the average cost per cycle. By a cycle we mean the following:

Let us begin at the moment our service station has been closed down - there are no customers waiting for service. Call phase I that time it takes for i customers to accumulate before the service station. By the second phase, is meant the time from when the ith arrival enters the system until the end of the present time period. Let the average number of arrivals in this phase be denoted by α'. Let the third phase consist of servicing these $i + \alpha'$ customers along with their generation arrivals (all arrivals occurring while the original $i + \alpha'$ are being served) until there is no queue and our service station is once again closed down. Phases I, II and III make up one cycle.

The average time for phase I is i / λ, the assembly time for i customers.

Let F(t) be the distribution of the length of time for phase II. Let E(t) be the expected length of time for phase II. Then,

$$\alpha' = \int_0^\infty \sum_{j=0}^{\infty} j e^{-\lambda t} \frac{(\lambda t)^j}{j!} \, dF(t) = \int_0^\infty \lambda t \, dF(t) = \lambda E(t)$$

and hence $E(t) = \frac{\alpha'}{\lambda}$.

Let,

$$\rho = \frac{\lambda}{\mu} \tag{1}$$

be the traffic intensity. In addition define P_b and P_c as the time of busy and clear periods for ordinary queuing systems. Then, because of the Poisson arrival process,

$$P_c = 1/\lambda \tag{2}$$

Also, the fraction of time the server is busy is given by

$$\rho = \frac{P_b}{P_b+P_c} \tag{3}$$

On combining (1), (2) and (3), we obtain,

$$P_b = \frac{1}{\mu-\lambda}$$

This is the expected length of the busy period for one customer and, hence, the expected duration of the third phase is $(i+\alpha')/(\mu-\lambda)$.

The average length of a cycle can then be easily obtained by summing,

$$T = \frac{i}{\lambda} + \frac{\alpha'}{\lambda} + \frac{(i+\alpha')}{\mu-\lambda} = \frac{i(\mu-\lambda)+\alpha'(\mu-\lambda)+(i+\alpha')\lambda}{\lambda(\mu-\lambda)}$$

$$= \frac{\mu(i+\alpha')}{\lambda(\mu-\lambda)} = \frac{i+\alpha'}{\lambda(1-\rho)} \tag{4}$$

Also, the fraction of time the system is in each of the phases is,

$$\frac{i}{\lambda T} = \frac{i(1-\rho)}{i+\alpha'} \tag{5}$$

$$\frac{\alpha'}{\lambda T} = \frac{\alpha'(1-\rho)}{i+\alpha'} \tag{6}$$

$$\frac{i+\alpha'}{(\mu-\lambda)T} = \rho \tag{7}$$

for phases I, II, and III respectively.

Let us adopt the following notation:

p_{kj} = Pr There are k customers in the system and there are j available servers

$K = 0, 1, \ldots$ and $j = 0, 1$

t_t-A random variable denoting the length of time from the ith

arrival to the end of the next observation point.

τ_s - A random variable denoting the time between the arrival of a customer in the third phase until the completion of the particular service in progress.

τ_t - A random variable denoting the time between the arrival of a customer in the second phase until the end of the second phase.

W - Average queuing time

L - Average queue length

Let us note some important relationships:

Since phase I is just the arrival of customers through a Poisson stream all states in phase I will occur with the same probability. Hence, from (5)

$$p_{K0} = \frac{1-\rho}{i+\alpha'}, \qquad K = 0, 1, \ldots, i-1 \tag{8}$$

Also, from (6) the probability of being in phase II is

$$\sum_{K=i}^{\infty} p_{K0} = \frac{\alpha'(1-\rho)}{i+\alpha'} \tag{9}$$

Finally, the associated probability for phase III is,

$$\sum_{K=1}^{\infty} p_{K1} = \rho \tag{10}$$

Next, Cox and Smith[4] show that the forward delay, τ, is distributed so that

$$E(\tau) = E(t)\left(\frac{1+\nu^2}{2}\right) \tag{11}$$

where t is the original random variable and ν, the coefficient of variation of t. For the exponential distribution, $\nu = 1$.

Now, we proceed to analyze the queuing time of customers arriving in different phases. For a customer who arrives in phase I, where there are K customers present, his waiting time is,

$$(i-1-K)/\lambda + E(t_t) + (K+1)/\mu$$

An arrival in the second phase when the system is in state K can expect to wait,

$$E(\tau_t) + (K+1)/\mu$$

Finally, a customer in the third phase, when K are present at his arrival, will wait on the average,

$$E(\tau_s) + K/\mu$$

Now we are in the position to write down the average queuing time W:

$$W = \sum_{K=0}^{i-1} p_{K0}[(i-1-K)/\lambda + E(t_t) + (K+1)/\mu)]$$

$$+ \sum_{K=i}^{\infty} p_{K0} [E(\tau_t) + (K+1)/\mu]$$

$$+ \sum_{K=1}^{\infty} p_{1j} [E(\tau_s) + K/\mu]$$

Insertion of (1), (8), (9), (10), and (11) yields,

$$W = \frac{1}{\lambda} [(L+1)\rho + \frac{i(i-1)(1-\alpha')}{2(i+\alpha')}$$

$$+ \frac{\alpha' i(1-\rho)}{i+\alpha'} + \frac{\alpha'^2(1-\rho)}{i+\alpha'} \frac{(1+\nu_\tau^2)}{2}]$$

From Little's [19] formula,

$$L = \lambda W$$

the average queue length is then,

$$L = \rho + \frac{\rho^2}{1-\rho} + \frac{i(i-1)}{2(i+\alpha')} + \frac{\alpha'}{i+\alpha'}\left[i+\alpha'\left(\frac{i+\nu_\tau}{2}\right)^2\right]$$

which is the average queue length of a simple queue, given by the first two terms (Pollczek-Khintchine), plus the additional queue length due to the intermittent operation of the server.

Optimization

Now it is possible to write down the cost function for the averaging criterion.

First, in one cycle a cost $\frac{A+B}{T}$ will be incurred due to set-up and shut-down of the server. Also we can expect a contribution of K(L) as the average holding cost per cycle. Finally, in one cycle there is savings due to the decreased use of the server. The server is being used only when in phase III and, hence, the cost incurred from its use is expected to be $\rho C/T$.

Therefore, the average cycle cost is

$$C_t(i) = K(L) + \frac{\lambda(1-\rho)(A+B)}{i+\alpha'} + \rho C/T$$

To find the optimal opening point, i, we may differentiate C_t with respect to i, set equal to zero, and solve for i (an integral value is naturally required). However, this is impossible as given, as it is necessary for the form of $K(\cdot)$ to be explicitly given.

A Specific Case

First, for the sake of example only, and second, to compare the results with some previously cited works, let us assume that the holding costs, $K(\cdot)$, are linear with respect to the averaging values. Then,

$$C_t(i) = K\cdot[\rho + \frac{\rho^2}{1-\rho} + \frac{i(i-1)}{2(i+\alpha')} + \frac{\alpha'}{i+\alpha'}[i+\alpha'(\frac{1+\nu_t^2}{2})]]$$

$$+ \frac{(A+B)\lambda(1-\rho)}{i+\alpha'} + \rho C/T$$

Differentiating,

$$\frac{\partial C_t}{\partial i} = \frac{K\cdot(i+\alpha')(i-\frac{1}{2}) - \frac{Ki}{2}(i-1)}{(i+\alpha')^2}$$

$$+ \frac{K\cdot(i+\alpha')\alpha'[K\alpha' i + K\alpha'^2(\frac{1+\nu_t^2}{2})]}{(i+\alpha')^2} - \frac{(A+B)\lambda(1-\rho)}{(i+\alpha')^2} = 0$$

or,

$$Ki^2 + 2K\alpha' i + K\alpha'[\alpha'-1-\alpha'\nu_t^2] - 2(A+B)\lambda(1-\rho) = 0$$

or,

$$i^2 + 2\alpha' i + [\alpha'(\alpha'-1-\alpha'\nu_t^2) - \frac{2(A+B)\lambda(1-\rho)]}{K} = 0$$

which has roots,

$$(-2\alpha' \pm \sqrt{4\alpha'^2 - 4[\alpha'(\alpha'-1-\alpha'\nu_t^2) - \frac{2(A+B)\lambda(1-\rho)]}{K}})/2$$

or,

$$-\alpha' \pm \sqrt{\alpha'^2 - [\alpha'(\alpha'-1-\alpha'\nu_t^2) - \frac{2(A+B)\lambda(1-\rho)]}{K}}$$

Simplifying we obtain,

$$- \alpha' \pm \sqrt{\frac{2(1-\rho)\lambda(A+B)}{K} + \alpha'(1+d\nu_t^{\,2})}$$

The positive root yields the minimum and, hence

$$\underline{i} = -\alpha' + \sqrt{\frac{2(1-\rho)\lambda(A+B)}{K} + \alpha'(1+\alpha\nu_t^{\,2})} \tag{12}$$

The optimal solution is then obtained by comparing the cost of the nearest positive integers $[\underline{i}]$, $[\underline{i}+1]$ and the possibility of permanent server availability. This latter cost is just

$$C_s = K\cdot[\rho + \frac{\rho^2}{1-\rho}] + C/T$$

That is, i* is defined by

$$C(i^*) = \min\ [C[\underline{i}],\ C([\underline{i}+1]),\ C_s]$$

(in the last case $i^* = 0$).

It is interesting to note that the value of $\underline{i}$ obtained by differentiation is not a function of the running cost, C, of the server. Actually, this result is not surprising as over a reasonable length of time the same number of customers will be served on the average; hence, the server will be running for the same length of time.

Finally, if the system is observed after each departure, rather than at equal intervals, phase II disappears. Therefore, (12) becomes

$$\underline{i} = \sqrt{\frac{(A+B)\lambda(1-\rho)}{K}}$$

This result agrees precisely with that of Heyman's.

Computational Experience Using Howard's Algorithm

Brief Description of Algorithm

Given a process which recieves a reward q_i at the beginning of a period when found in state i, then, if $V_i(n)$ is the present value of the total expected return when in state i with n transitions remaining, we obtain

$$V_i(n) = q_i + \alpha \sum_{j=1}^{M} [p_{ij} V_j(n-1)]$$

for each i and n where α is some discount factor and a total of M states.

With the introduction of alternatives, Howard[9] has developed an algorithm which minimizes these present values for an infinite horizon problem, i.e., replacing $V_i = \lim_{n\to\infty} V_i(n)$ for $V_i(n)$. This routine can be summarized by the following flowchart in Figure 2.

The iteration process can be entered in either box and Howard guarantees that each cycle will yield a present value at least as good as the one previously obtained. He also guarantees that there is convergence to an optimal policy [1] and this is noted by having policies on two successive iterations identical.

In the problem at hand either of two decisions may be made in any of the 2M states. The states are characterized

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[1] Veinott [16] has guaranteed convergence in a finite number of steps.

<u>Value Determination Operation</u>

Use the p_{ij} and the q_i for the given policy to solve the set of equations

$$V_i = [q_i + \alpha \sum_{j=1}^{N} p_{ij} V_j] \quad i = 1,2,\ldots,N$$

for all present values V_i.

<u>Policy Improvement Routine</u>

For each state i, find the alternative k* that minimizes

$$[q_i^k + \alpha \sum_{j=1}^{N} p_{ij}^k V_j]$$

using the present values V_i from the previous policy. Then k* becomes the new decision in the ith state, p_{ij}^{k*} becomes p_{ij} and q_i^{k*} becomes q_i.

Figure 2: Policy Interation with Discounting

by the number in the system 0, 1, ..., M-1 and whether the server is closed or opened. The cycle was entered in the policy improvement routine by choosing all the present values as zero. The first step, therefore, was minimizing immediate returns, i.e., choose the alternative that chooses

$$q_i = \min \begin{Bmatrix} K(i) \\ K(i) + A + C \end{Bmatrix}$$

when the server is closed and

$$q_i = \min \begin{Bmatrix} K(i) + B \\ K(i) + C \end{Bmatrix}$$

when the server is opened. The new present values were then calculated and iteration process continued until convergence was obtained.

Calculation of Probabilities

In order to make use of the algorithm explicit values for the transition probabilities, p_{ij} and q_{ij} had to be known.

When the server was closed, p_{ij}, the probability of going from i to j in one time period, was simply the probability of j-i arrivals, where the arrivals are Poisson distributed with parameter λ.

With the server opened, the probabilities q_{ij} are derived as follows. Prabhu[14] finds the Pr (there are $\leq j$ in the system at time t | there are i in the system at time 0)

$$= \Pr(Q(t) \leq j \mid Q(0) = i)$$

$$= \sum\sum_{\Delta(j-i)} \frac{e^{-(\lambda+\mu)t}\lambda^a\mu^b t^{a+b}}{a!b!} - \sum\sum_{\Delta(-j-i-2)} \frac{e^{-(\lambda+\mu)t}\lambda^{a+j+1}\mu^{b-j-1}t^{a+b}}{a!b!}$$

where $\Delta(j)$ denotes the set of integers a, b such that $a>0$, $b>0$ and $a-b\leq j$.

Subtracting from this quantity

$$\Pr\ (Q(t) \leq j-1 \mid Q(0) = i)$$

we get

$$\Pr\ (Q(t) = j \mid Q(0) = i)$$

Next, setting t=1, i.e., one time period, we establish

$$q_{ij} = e^{(\lambda+\mu)}\{ \sum\sum_{a-b=j-i} \frac{\lambda^a\mu^b}{a!b!} + \sum\sum_{\Delta(-j-i-1)} \frac{\lambda^a\mu^b}{a!b!} - \sum\sum_{\Delta(-j-i-2)} \frac{\lambda^{a+j+1}\mu^{b-j-1}}{a!b!}\}$$

Since we always make the decision to open at state M and above, these probabilities are truncated at M. This is necessary as Howard's routine allows for only a finite number of states.

Results

A FORTRAN program was used to calculate these probabilities and then incorporated into another which was Howard's policy improvement routine. Some of the results obtained were both reassuring and enlightening. Since, however, sensitivity on one parameter was accomplished while the other parameters were fixed at particular levels any definite conclusions were difficult to reach.

Sensitivity to discount factor

Theorem 5 in Chapter II stated that it was possible to close the server with customers present for the infinite

TABLE 1

SENSITIVITY OF DISCOUNT FACTOR TO HOLDING COST FOR $\lambda=1.0$, $\mu=2.0$, A=10, B=10 AND C=40

α	j* for K(X) =		
	X	$X^2 + X$	e^X
0.9	0	0	0
0.8	0	0	0
0.7	M	0	0
0.6	M	0	0
0.5	M	2	0
0.4	M	4	1
0.3	M	M	2
0.2	M	M	3
0.1	M	M	4

horizon discounted case. For fixed costs, however, this was only true for small α. Table 1 gives j*, the closing point as a function of three different parameter values. Notice that the values are not only small but also as the holding costs became "more convex" the value of the discount factor when the server is closed with customers present decreases.

Other results

Certainly extreme cases led to extreme results as far as determination of i* and j*. For example, for high set-up and shut-down costs and low operating costs as soon as we open we always remain open. Also, if the holding cost is small compared to the fixed costs once we close, we will always remain closed (since the fixed costs are constants, this only occurred in our investigation when the holding cost was linear).

An interesting observation was that the sensitivity of i* to the ratio of the arrival and service rates (λ/μ) was very slight. However, i* was quite sensitive to the magnitude of λ and μ. When λ and μ were chosen large (≥ 1) i* turned out to be only 0 or 1. However, when their magnitudes were decreased ($\lambda=0.1$, $\mu=0.2$) i* had the value 4. Certainly there is variation here depending on costs but large arrival and service rates imply larger shifts in state. This means having large holding costs and soon after no holding costs and conversely. This implies many set-up and shut-downs and hence i* is very low so that the server would only be shut down when empty.

CHAPTER IV

MULTI-SERVER MODEL

Introduction

Let us return now to the more general problem, that of finding optimal policies for the operation of a service mechanism with an arbitrary number of servers.

The assumptions made at the beginning of Chapter II will still hold, except the following change:

(I') Assume there are s servers. Their service times are independent exponential random variables each with rate μ. Customers are still processed in the order of their arrival. If a server is closed down whil processing a customer, that customer rejoins the queue.

Also, just as in the single server case, we shall assume that for k, the number of open servers, our arrival rate is such, that given any $\varepsilon>0$, there exists and $M(\varepsilon)$ such that:

$$P[X_t>M(\varepsilon)]<\varepsilon$$

This along with the assumption that $s\lambda<\mu$ will guarantee that decisions will only have to be made in a finite number of states.

The cost structure is affected only in that there is now an operating cost, set-up cost and shut-down cost for each server, respectively C, A, and B. The transition probabilities

will now be denoted as

$$p_{ij}^{k} = \Pr\{X_{t+1} = j \mid X_t = i \text{ and } k(=0,1,\ldots, s) \text{servers are opened}\}$$

where X_t is again the number of customers in the system at time t.

Form of the Optimal Policy

In this section we will describe a policy which we will define to be optimal. This policy most closely resembles that of the (j*, i*) for the single server.

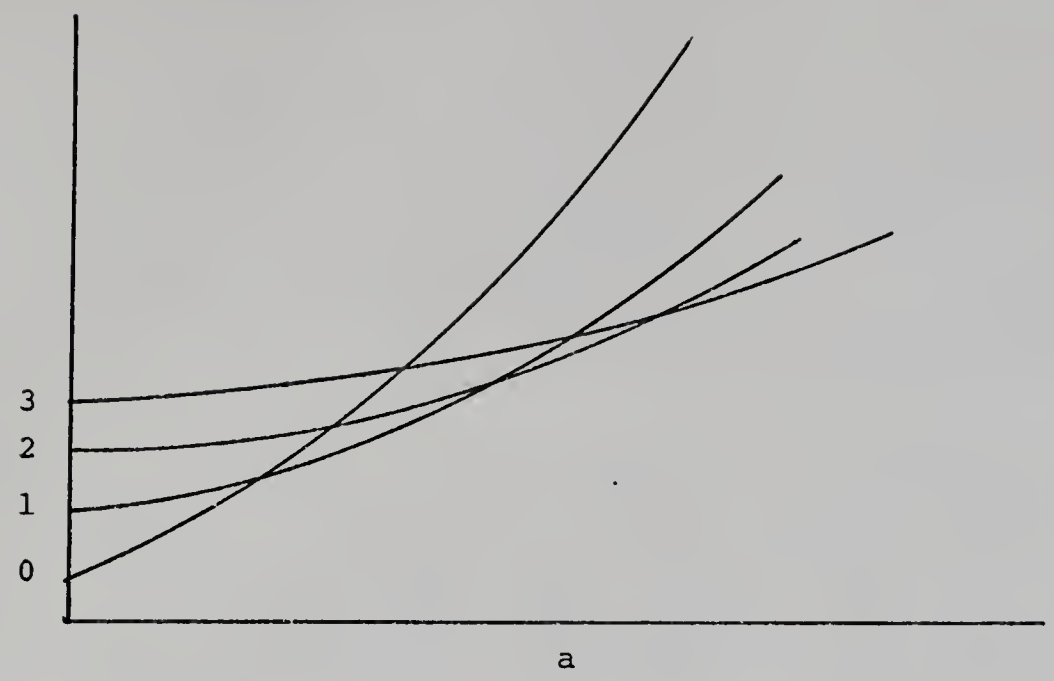
Given the state of the system, we are interested in determining unique numbers which tell us how many servers will be open after a decision is made. Notationally, let us describe an optimal policy as follows:

$$(i,k : j_0^*, j_1^*, \ldots, j_{k-1}^*; i_{k+1}^*, \ldots, i_s^*) \qquad (1)$$

where the i, k represents the present state of the system, i.e., i customers present and k servers opened. The sequences $\{j_\ell^*\}$ and $\{i_\ell^*\}$ are nondecreasing sequences with the following interpretation:

If $i_{m-1}^* \leq i < i_m^*$ and $k \leq m-1$, open m-1-k servers (have m-1 open after decision). If $j_{m-1}^* < i \leq j_m^*$ and $k \geq m$, close k-m servers (have m open after the decision). If $i \geq i_s^*$ or $i \leq j_0^*$, have all the servers open or closed respectively. If i does not meet any of these conditions keep k servers open.[1]

[1] Note how this conforms to the original form for an optimal policy described in Chapter I.



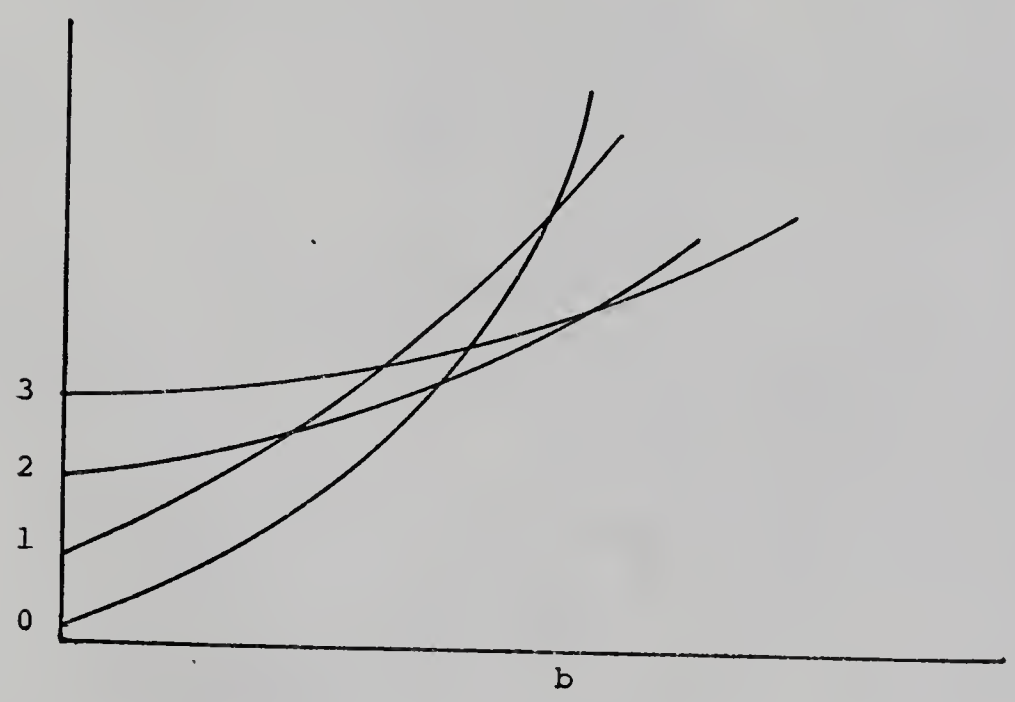


Figure 3: Costs associated with a multi-server queue.

An optimal policy may permit going from j to j+1 or j to j-1 for all j. If this is the case both sequences are increasing. However, an optimal policy may dictate a state ℓ where the decision to open only <u>one</u> server will never be made. In this case $i^*_{\ell+1} = i^*_\ell$ (or, possibly $j^*_{\ell-1} = j^*_\ell$). This is best illustrated by a simple example.

Let us assume that s=3 and presently i=0. Figure 3 depicts two possibilities as to what the cost might be as a function of the number of servers to be opened.

In (a), i^*_1, i^*_2 and i^*_3 are well defined to be the crossing points of the (0,1), (1,2) and (2,3) intersections respectively; and the sequence of i's is increasing. In (b) it is never optimal to have one server open so in order to conform to our rules our optimal policy will have $i^*_1 = i^*_2$ and the sequence of i's is nondecreasing.

Dynamic Programming Formulation

Now that the elements of the model are given let us again set up the dynamic programming formulation for the finite horizon.

Define $\phi_n(i,\alpha,k)$ as the minimum expected cost with i customers present, k servers opened with n periods remaining in our horizon, using discount factor α.

Then our recursive relationships are:

$$\phi_n(i,\alpha,k) = \min \begin{bmatrix} K(i)+kB & +\alpha \Sigma\, p_{ij}^{0}\phi_{n-1}(j,\alpha,0) \\ K(i)+(k-1)B+C & +\alpha \Sigma\, p_{ij}^{1}\phi_{n-1}(j,\alpha,1) \\ \vdots & \vdots \\ K(i)+kC & +\alpha \Sigma\, p_{ij}^{k}\phi_{n-1}(j,\alpha,k) \\ K(i)+(k+1)C+A & +\alpha \Sigma\, p_{ij}^{k+1}\phi_{n-1}(j,\alpha,k+1) \\ \vdots & \vdots \\ K(i)+sC+(s-k)A & +\alpha \Sigma\, p_{ij}^{s}\phi_{n-1}(j,\alpha,s) \end{bmatrix} \qquad (2)$$

where

$$\phi_0(i,\alpha,k) = K(i)$$

and as an analogous assumption from the single channel system we can say that there exist numbers M_0, M_1, ... M_{s-1} which demand that

$$\phi_n(m,\alpha,k)=K(m)+(k+1)C+A+\alpha \Sigma\, p_{mj}^{k+1}\phi_{n-1}(j,\alpha,k+1) \text{ for } m\geq M_k \qquad (3)$$

where there is a one-to-one correspondence between the k and M_k. Since this sequence $\{M_k\}$ is finite there is certainly a maximum and for simplicity we will impose (3) for $m \geq \max\{M_k\}=M$. All this says that whenever M customers are reached and k servers are opened we will automatically open one more server.

Furthermore, the assumptions that K(i) is nondecreasing and convex will again be made, along with assuming that for each nondecreasing function h(j)

$$\sum_{j=0}^{\infty} p_{ij}^{k} \quad h(j)$$

is nondecreasing in i for each k.

Proof for Finite Horizon

The same sequence of developing is used for the multi-server system as was used for single server. In this chapter only this logical sequence of steps will be mentioned to ensure our results for an arbitrary number of servers. Proofs for the theorems will appear in Appendix 2.

For n=1, let us illustrate how the result carries by considering s=2. Then,

$$\phi_1(i,\alpha,0) = \min \left\{ \begin{array}{l} K(i) \quad +\alpha \Sigma p_{ij}^0 K(j) \\ K(i)+A+C +\alpha \Sigma p_{ij}^1 K(j) \\ K(i)+2(A+C)+\alpha \Sigma p_{ij}^2 K(j) \end{array} \right\}$$

$$\phi_1(i,\alpha,1) = \min \left\{ \begin{array}{l} K(i) + B +\alpha \Sigma p_{ij}^0 K(j) \\ K(i)+C+\alpha \Sigma p_{ij}^1 K(j) \\ K(i)+A+2C+\alpha \Sigma p_{ij}^2 K(j) \end{array} \right\}$$

$$\phi_1(i,\alpha,2) = \min \left\{ \begin{array}{l} K(i) + 2B + \alpha \Sigma p_{ij}^0 K(j) \\ K(i)+B+C+\alpha \Sigma p_{ij}^1 K(j) \\ K(i)+2C +\alpha \Sigma p_{ij}^2 K(j) \end{array} \right\}$$

when the server is closed it must be shown that

$$\Sigma K(j)[p_{ij}^0 - p_{ij}^1] \text{ is nondecreasing}$$

and that

$$\Sigma K(j)[p_{ij}^1 - p_{ij}^2] \text{ is nondecreasing}$$

(The sum of these two is just

$$\Sigma K(j)[p_{ij}^{0} - p_{ij}^{2}]$$

which would certainly be nondecreasing as it is the sum of two nondecreasing functions.) Doing this leads to proving Theorem 1: For n=1 the policy of form (1) is optimal.

To extend the results for arbitrary n, let us point out the steps that make the analysis for the single server and that for an arbitrary number of servers similar.

For n=2 it becomes necessary to show that

$$\phi_1(i+x - \sum_{i=0}^{k_1} y_i, \alpha, k_1) - \phi_1(i+x - \sum_{i=0}^{k_2} y_i, \alpha, k_2)$$

for each k_1 and k_2, such that $k_2 > k_1$, is a nondecreasing function of i. This is equivalent to saying that there is at most one crossing point for any two alternatives and since the curves are nondecreasing (1) will be ensured optimal. The proof for this is the proof of Theorem 2 in the appendix and is analagous to Theorem 2 of Chapter II.

Again, continuing with the induction hypothesis, i.e., assuming $\phi_n(i,\alpha,k)$, n=3,4,..., k has an optimal rule given by (1), the enumeration procedure becomes more difficult but once again Theorem 2 of Chapter II follows in precisely the same manner (this once again is so because alternatives only have to be compared two at a time). This is done first, by providing the analog of Lemma 3 (proving $i_{k+1}^{*} \geq j_{k-1}^{*}$), and secondly, by using the transitive relations that if a-b is nondecreasing and b-c is nondecreasing, then a-c is nondecreasing (a-c=(a-b)+(b-c)). The details are in Appendix II.

Also, the discussion on piecewise convexity is simply extended. The definition allows a function to be the smallest of a denumerable number of convex functions and, hence, the addition of the extra alternatives still permits $\phi_n(i,\alpha,k)$ to be piecewise convex. Also, the argument related to the terminal points holds since having the possibility of s servers just allows for the possibility of s different weights.

This completes the discussion for the finite horizon and we have shown that

Theorem 2: For an arbitrary number of servers our optimal policy for the cost functions given in (2) is satisfied by (1).

Infinite Horizon and Averaging

Proceeding as before we shall prove that (1) is the optimal policy for the infinite horizon, i.e., when

$$\phi(i,\alpha,k) = \begin{bmatrix} K(i) + BK \quad +\alpha \Sigma p_{ij}^{0} \phi(j,\alpha,0) \\ K(i)+B(k-1)+C+\alpha \Sigma p_{ij}^{1} \phi(j,\alpha,1) \\ \vdots \\ K(i) + kC +\alpha \Sigma p_{ij}^{k} \phi(j,\alpha,k) \\ K(i) + (k+1)C+A+\alpha \Sigma p_{ij}^{k+1} \phi(j,\alpha,k+1) \\ \vdots \\ K(i) + sC + (s-k)A+\alpha \Sigma p_{ij}^{s} \phi(j,\alpha,s) \end{bmatrix} \tag{4}$$

Again the proof will be done for only one state (i,k) but by its nature is seen to hold for all states. The proof is Theorem 3 in Appendix II.

Moving into the discussion on the averaging criterion, the assumption that for each number of open servers there are

only a finite number of decision points we are able to prove that (1) is the optimal policy as shown in Theorem 4 from Chapter II. Theorem 4 in Appendix II gives this proof implying that (1) is the optimal rule for $\phi(i,k)$ where

$$\phi(i,k) = \lim_{s\to\infty} (1-\alpha_s)\phi(i,\alpha_s,k) \text{ where}$$

$$\phi(i,k) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} g(i,k) \text{ and } g(X_t,k) \text{ is the one step loss}$$

function.

Concluding Remarks

Certainly, the extension of Theorem 5 in Chapter 2 holds for the multi-server model. That is, we will never close all the servers with customers present, except possibly when there is a finite horizon.

In essence, therefore, this chapter along with Appendix II shows that the multi-server and single server model are very similar, in the sense of choosing optimal policies. This is, however, an exception rather than the rule in dynamic programming formulations. Bellman points out that under very general conditions when there are only two possible decisions that there is some point which uniquely separates our decision region. However, for more than two decisions he maintains that in general our decision regions are not so divided.[2]

[2] Bellman, R., op.cit., page 75.

CHAPTER V

CONCLUSION AND EXTENSIONS

The purpose of this paper was to ascertain optimal policies for queuing systems. The form of the optimal policy for the single server had been found before, but only when observations were at the end of arrival and departure epochs, rather than having observations in equal time intervals. Also, the only work previously done in existence of optimal policies for multi-server models was a two-server case discussed by Heyman. In this discussion, he always had one server open and the policy was to determine when to open the other. We allow for any finite number of servers. We are able to conclude that the number of servers open at some point in time is given by an intuitively appealing rule. This rule is analagous to the (s,S) policy in inventory theory.

The applications of the work in this paper are quite widespread. In any situation where there is a variable number of service stations which may be open, one would know how many servers should be opened or closed. More might be learned by observing additional computational results, as the purpose of this paper is not to find optimal policies but only the form at which they take on. Sobel points out many specific

problems relating optimal policies for queuing systems to inventory and replacement theory.

There are several extensions to be seen from the work done in this paper. Certainly there is a need for greater computational facility in obtaining the optimal policy for a given model. At present the closest computational experience for the multi-server model is in a paper by Moder and Phillips.[12] Also, there is no reason why Howard's scheme cannot be extended to deal with this case. Computational experience is very important for a problem of this type as its application will only become meaningful when the parameters which we might be free to choose are set at the proper levels. That is to say that even though our optimal policy is determined only from the number in the system and the number of servers open, there may be other parameters, such as service rate, which might substantially affect our optimal policy.

The main restriction that had to be made, so that periodic review can be used, was that our arrival and service distributions were exponential. As was mentioned this assumption could be dropped if we are willing to increase our state space. That is, the state would be not just a number in the system and the number of open servers but also the time from the last arrival, if arrivals were to become general. Investigation of this problem is to be considered. Computationally, one would almost certainly have to use Jewell's Markov-renewal technique.

Another extension is to consider finding optimal policies when there are several machines with different service rates.

This was avoided because one must either make a predetermined rule, or include the finding of the rule as part of finding the optimal policy to determine which of the available machines an arriving customer will be serviced on. When this is done one can introduce switching type policies as alternatives, i.e., if a server is closed down while serving a customer, the customer may immediately switch to another machine. Heyman discusses rules of this type.

A slightly different but very interesting problem can be thought of as an extension is that of finding the optimal policy as a function of some other parameter. In particular, finding an optimal rule as a function of the arrival rate, rather than of the number in the system, may have useful applications. If one could check on the rate at which customers entered a supermarket, for example, it might be possible to predict when servers should be opened or closed so as to avoid idle time and long queues.

Of course, balking or reneging can also be introduced to the system. In connection with this we could consider a modified loss system which has application to telephone switchboards. That is, consider a customer who tries to enter the system and is turned away (placing a call and finding all circuits busy) but will return with a certain probability. Associating a cost with a lost customer it is to be determined how many servers should be kept open so as to minimize cost.

Another change in the model would be to change the queue discipline; introduction of priorities may certainly make this work applicable to time-shared computer systems.

APPENDIX I

PIECEWISE CONVEXITY

The following is due to Zangwill[18].

Definition: Piecewise convex function (PCF) -- A function $f(Z)$ is said to be piecewise convex if there is a sequence of convex functions $\{g_i(Z)\}$ such that

$$f(Z) = \inf_i \{g_i(Z)\}$$

Definition: Let $f(Z) = \inf_i \{g_i(Z)\}$ be PCF. Define a <u>basic set</u> B_i as

$$B_i = \{Z \mid f(Z) = g_i(Z)\}$$

Definition: Let the <u>terminal set</u> T be defined as

$$T = \{Z \mid f(Z) = g_i(Z) = g_k(Z) \quad \text{for } i \neq k\}$$

The next lemma is an important one in the concept of PCF.

Lemma 1: Let $\{f_i(Z)\}$ be a sequence of PCF each with respective terminal set T_i. Then $F(Z) = \sum_i w_i f_i(Z)$, with $w_i > 0$ is PCF with terminal set $T = \bigcup_i T_i$.

Proof: The proof is by induction on the number of functions. For one, the result is clearly true. Assume it is true for k, i.e.,

$$F_k(Z) = \sum_{i=1}^{k} w_i f_i(Z) \text{ is PCF with terminal set } T^k = \bigcup_{i=1}^{k} T_i$$

By the definition of PCF

$$F_k(Z) = \inf_i \{g_i^1(Z)\}$$

for some sequence of convex functions.

Let $w_{k+1}f_{k+1}(Z) = \inf_j\{g_j^2(Z)\}$ for a sequence of convex functions. Then

$$F_{k+1}(Z)=F_k(Z)+w_{k+1}f_{k+1}(Z)= \inf_i\{g_i^1(Z)\}+\inf_j\{g_j^2(Z)\}$$

$$= \inf_{i,j}\{g_i^1(Z) + g_j^2(Z)\}$$

But $g_i^1(Z) + g_j^2(Z)$ is convex for all i and j and hence $F_{k+1}(Z)$ is PCF.

Now, a point h is in the terminal set of $g^1(\cdot) + g^2(\cdot)$ if and only if $g^1(h) + g^2(h) = g_{i_1}(h)+g^2_{j_1}(h)=g^1_{i_2}(h)$ and either (a) $i_1 \neq i_2$ and $j_1=j_2$, (b) $i_1=i_2$ and $j_i \neq j_2$, or (c) $i_1 \neq i_2$ and $j_1 \neq j_2$. If (a) occurs then $h \in T^k$, if (b) occurs then $h \in T_{k+1}$ and if (c) occurs then $h \in T^k \cap T_{k+1}$. Also, if $h \in T^k$, then either (a) or (c) occurs; if $h \in T_{k+1}$, then either (b) or (c) occurs and $h \in T^{k+1}$. Another useful lemma is,

Lemma 2: If g(Z) and h(Z) are PCF then f(Z) = min [g(Z),h(Z)] is PCF.

Proof: By definition of PCF

$$g(Z) = \inf_i \{g_i(Z)\} \text{ and } h(Z) = \inf_j \{h_j(Z)\}$$

for some sequence g and h.

Hence,

$$f(Z) = \min\left[\inf_i \{g_i(Z)\},\ \inf_j \{h_j(Z)\}\right]$$

$$= \min\left[\inf_{i,j} \{g_i(Z)\ ,\ h_j\ (Z)\}\right]$$

$$= \inf_{i,j} \{g_i(Z),\ h_j(Z)\}$$

and $f(Z)$ is PCF.

Lemma 3: $\sum_{j=0}^{\infty} p_{ij}K(j)$ and $\sum_{j=0}^{\infty} q_{ij}K(j)$ are convex.

Proof: (Two ways are illustrated -- one for $\Sigma\, p_{ij}K(j)$ and the other for $\Sigma\, q_{ij}K(j)$)

$$\sum_{j=0}^{\infty} p_{ij}K(j) = E_x\ K(i+x)$$

$$= \sum_x P_r\ (x \text{ arrivals in time period})\ K(i+x)$$

which is a linear combination of convex functions and hence is convex.

Or as an alternative,

$$\theta K(i_1+x-y)+(1-\theta)K(i_2+x-y) \geq K(\theta i_1+(1-\theta)i_2+x-y)$$

$$\theta E_xE_yK(i_1+x-y)+(1-\theta)E_xE_yK(i_2+x-y) \geq E_xE_yK(\theta i_1+(i-\theta)i_2+x-y)$$

and hence

$$\sum_{j=0}^{\infty} q_{ij}\ K(j) = E_xE_y\ K(i+x-y)$$

is convex.

Recalling that,

$$\phi_1^0(i,\alpha) = \min \left\{ \begin{array}{l} K(i) + \alpha \sum_{j=0}^{N} p_{ij}\ K(j) \\ \\ K(i) + A + C + \alpha \sum_{j=0}^{N} q_{ij}\ K(j) \end{array} \right\}$$

and using the definition of PCF we conclude that ϕ_1^0 is PCF.

Next, since

$$\sum_{j=0}^{\infty} p_{ij}\ \phi_1^0(j,\alpha) = E_x\phi_1^0(i+x,\alpha) = \Sigma\ \Pr(x \text{ arrivals})\ \phi_1^0(i+x,\alpha)$$

and

$$\sum_{j=0}^{\infty} q_{ij}\ \phi_1^0(j,\alpha) = E_x E_y\ \phi_1^0(i+x-y,\alpha)$$

$$= \Sigma\ \Sigma\ \Pr(x \text{ arrivals})\ \Pr(y \text{ services})\phi_1^0(i+x-y,\alpha)$$

along with Lemma 1 enables us to say that $\Sigma\, p_{ij}\ \phi_1^0(j,\alpha)$ and $\Sigma\, q_{ij}\ \phi_1^0(j,\alpha)$ are PCF. Since the addition of a convex function to a piecewise convex function is obviously piecewise convex, $\phi_2^0(i,\alpha)$ is a PCF, a result achieved from Lemma 2.

Finally, alternate use of Lemmas 1 and 2 leads to the following:

Lemma 4: $\phi_n^0(i,\alpha)$ for all $n \geq 1$ is PCF.

The result also follows immediately for $\phi_n^1(i,\alpha)$.

Clearly, since the set of terminal points does not depend on the weights,

$$\Sigma\, p_{i+x,j}\ \phi_{k-1}^0\ (j,\alpha) \text{ and } \Sigma\, p_{i+x-y,j}\ \phi_{k-1}^0\ (j,\alpha)$$

have the same terminal points. Also, each interval between any two terminal points of a piecewise convex function is convex. Hence, to show

$$\Sigma[p_{i+x,j}\ \phi_{k-1}^0(j,\alpha) - p_{i+x-y,j}\ \phi_{k-1}^0\ (j,\alpha)]$$

is nondecreasing between terminal points one can resort to the discussion used to show the same thing for

$$\Sigma\ [p_{ij} - q_{ij}]\ K(j).$$

Repeated use of the fact that the terminal points are the same and that the expression is nondecreasing between these points enables us to conclude,

Lemma 5: $\Sigma\, [p_{i+x,j} - p_{i+x-y,j}]\ \phi^{0}_{k-1}\ (j,\alpha)$ is nondecreasing in i. Most certainly the result also holds for

$$\Sigma\, [q_{i+x,j} - q_{i+x-y,j}]\ \phi^{1}_{k-1}\ (j,\alpha)$$

APPENDIX II

EXTENSIONS OF THEOREMS TO MULTI-SERVER QUEUE

Lemma: $\sum_{j=0}^{\infty} [p_{ij}^{m} - p_{ij}^{n}] \; K(j)$ is nondecreasing for $m<n$.

Proof: Just as $\sum_{j} q_{ij} \; K(j) = E_x E_y \; K(i+x-y)$,

$$\sum_{j} p_{ij}^{m} \; K(j) = E_x E_{y_1} \ldots E_{y_m} \; K(i+x-\sum_{j=1}^{m} y_j)$$

Letting $\sum_{j=0}^{m} y_j = Y_m$, from Chapter II we know that

$K(i+x-Y_m) - K(i+x-y_n)$ for $Y_n > Y_m$ is nondecreasing

in i. Hence, so would the difference in their expectations be and

$$\sum_{j=0}^{\infty} [p_{ij}^{m} - p_{ij}^{n}] \; K(j)$$

for all $m<n$ is nondecreasing.

Theorem 1: For n=1 the optimal policy is of form (1) in Chapter IV.

Proof: Let k be the number of open servers.

Then,

$$\phi_1(i,k,\alpha) = \min \begin{bmatrix} K(i)+Bk & +\alpha \Sigma p_{ij}^{0} K(j) \\ K(i)+B(k-1)+ & C+\alpha \Sigma p_{ij}^{1} K(j) \\ \vdots \quad \vdots & \vdots \\ K(i)+ & kC+\alpha \Sigma p_{ij}^{k} K(j) \\ K(i)+A & +(k+1)C+\alpha \Sigma p_{ij}^{k+1} K(j) \\ \vdots \quad \vdots & \vdots \\ K(i)+A(s-k)+ & \alpha \Sigma p_{ij}^{s} K(j) \end{bmatrix}$$

The policy (1) implies that there exist unique numbers which determines exactly how many servers should be open with i customers in the system, given k were open. Now, for any two alternatives m and n, m<n.

$$K(i) + (\text{Constant}) + \alpha \Sigma p_{ij}^{m} K(j) \left(\begin{smallmatrix} < \\ \geq \end{smallmatrix}\right)$$

$$K(i) + (\text{Constant})' + \alpha \Sigma p_{ij}^{n} K(j)$$

or

$$\Sigma [p_{ij}^{m} - p_{ij}^{n}] \quad K(j) = \underline{\text{Constant}}$$

can occur for at most one value of i. Since the left side is nondecreasing from the preceding lemma, once alternative n is chosen, alternative m can never be chosen again for all m<n. Hence, the sequences are nondecreasing, which determines when an alternative should be chosen.

Lemma: $\Sigma [p_{ij}^{k_1} \phi_1 (j,\alpha,k_1) - p_{ij}^{k_2} \phi_1 (j,\alpha,k_2)$ is nondecreasing for $k_1<k_2$

Proof: This can be written as

$$E[\phi_1(i+x-Y_{k_1},\alpha,k_1) - \phi_1(i+x-Y_{k_2},\alpha,k_2)]$$

where the expectation is taken over random variables x, Y_1, Y_{k_2}. If the bracket is nondecreasing for all i then certainly the expectation will be. So let us consider

$$\phi_1(i+x-Y_{k_1},\alpha,k_1) - \phi_1(i+x-Y_{k_2},\alpha,k_2)$$

Now for any i,

$$\phi_i(i+x-Y_{k_1},\alpha,k_1 0 = K(i+x-Y_{k_1}) + (\text{Constant})$$

$$+\alpha \Sigma\, p_{i+x-Y_{k_1}},K(j)$$

The difference is nondecreasing in i since $K(i+x-Y_{k_1})$ - $K(i+x-Y_{k_2})$ is nondecreasing and the difference of the summations is by the first lemma in the appendix.

Assuming that $\phi_n(i,\alpha,k)$ n=2,3,4,...r has an optimal rule given by (1) and that

$$\Sigma\, [p_{ij}^{k_1}\ \phi_n\ (j,\alpha,k_1) - p_{ij}^{k_2}\ \phi_n\ (j,\alpha,k_2 0] \text{ for } k_1 < k_2$$

is nondecreasing in i, (1) must be shown optimal for n=r+1.

Theorem 2: (1) is optimal for ϕ_{r+1}

Proof: Since

$$\phi_{r+1}(i,\alpha,k) = \left\{ \begin{array}{ll} K(i)+kB & +\alpha \Sigma\, p_{ij}^{0}\ \phi_r(i,\alpha,0) \\ \vdots \quad \vdots & \vdots \\ K(i)+ & kC+\alpha \Sigma\, p_{ij}^{k}\ \phi_r(i,\alpha,k) \\ \vdots \quad \vdots & \vdots \\ K(i)+(s-k)A+sC & +\alpha \Sigma\, p_{ij}^{s}\ \phi_r(i,\alpha,s) \end{array} \right\}$$

for optimal solution to be (1) it must again be true for alternatives m and n, m < n, the difference must be

nondecreasing. This will guarantee that the sequences $\{j^*\}$ and $\{i^*\}$ will be nondecreasing. Since this difference is

$$\Sigma\,[p_{ij}^{m}\,\phi_r(i,\alpha,m) - p_{ij}^{n}\,\phi_r(i,\alpha,n)]+(\text{constant})$$

it is nondecreasing by the induction assumption.

Theorem 3: $\phi(i,\alpha,k)$, as defined by (4) in Chapter IV, has an optimal policy given by (1) in Chapter IV.

Proof: (Done for an arbitrary state (i,k))

Define $T_j(\phi_{n-1})$ as the cost associated with the jth alternative $j=0,1,\ldots,s$. For example,

$$T_0(\phi_{n-1}) = K(i) + Bk + \alpha\,\Sigma\, p_{ij}\,\phi_{n-1}(j,\alpha,0)$$

Then,

$$\phi_n(i,\alpha,k) = \min_{j=0,\ldots,s} T_j\,(\phi_{n-1})$$

Let $j(n)$ = index which gives the minimum of $T_j\,(\phi_n)$, $j=0,\ldots,s$.

This implies that

$$\phi_n(i,\alpha,k) = T_{j(n-1)}(\phi_{n-1}) \leq T_{j(n-1)}(\phi_n)$$

and

$$\phi_{n+1}(i,\alpha,k) = T_{j(n)}(\phi_n) \leq T_{j(n-1)}(\phi_n)$$

Subtracting,

$$|\phi_n(i,\alpha,k)-\phi_{n+1}(i,\alpha,k)| \leq \max\,[|T_{j(n-1)}(\phi_{n-1})-T_{j(n-1)}(\phi_n)|,$$
$$|T_{j(n)}(\phi_n)-T_{j(n)}(\phi_{n-1})|]$$

$$\leq \max_{j=0,1,\ldots,s} [|T_j(\phi_{n-1}) - T_j(\phi_n)|]$$

$$\leq \max \Big[|K(i)+kB+\alpha \Sigma\, p_{ij}^{0}\, \phi_{n-1}(j,\alpha,0) -$$

$$(K(i)+kB+\alpha \Sigma\, p_{ij}^{0}\, \phi_n(j,\alpha,0))|, \ldots$$

$$|K(i)+Ck+\alpha \Sigma\, p_{ij}^{k}\, \phi_{n-1}(j,\alpha,k) -$$

$$(K(i)+Ck+\alpha \Sigma\, p_{ij}^{k}\, \phi_n(j,\alpha,k))|, \ldots$$

$$|K(i)+sC+(s-k)A+\alpha \Sigma\, p_{ij}^{s}\, \phi_{n-1}(j,\alpha,s) -$$

$$(K(i)+sC+(s-k)A+\alpha \Sigma\, p_{ij}^{s}\, \phi_n(j,\alpha,s))| \Big]$$

$$\leq \max_{r} \quad \alpha |\Sigma\, p_{ij}^{r}\, (\phi_{n-1}(j,\alpha,r) - \phi_n(j,\alpha,r))|$$

$$\leq \alpha \Sigma\, p_{ij}^{k} |\phi_{n-1}(j,\alpha,k) - \phi_n(j,\alpha,k)|$$

Hence

$$\max_i |\phi_n(i,\alpha,k) - \phi_{n+1}(i,\alpha,k)| \leq$$

$$\alpha \Sigma\, p_{ij}^{k} \max_i |\phi_{n-1}(i,\alpha,k) - \phi_n(i,\alpha,k)|$$

$$\leq \alpha \max_i |\phi_{n-1}(i,\alpha,k) - \phi_n(i,\alpha,k)|$$

Define, $U_n = \max_i [|\phi_n(i,\alpha,k) - \phi_{n-1}(i,\alpha,k)|]$

and hence

$$U_{n+1} \leq \alpha U_n \leq \alpha^2 U_{n-1} \ldots \leq \alpha^n U_1$$

and, since

$$\sum_{n=0}^{\infty} U_{n+1} \leq \sum_{n=0}^{\infty} \alpha^n U_1 = \frac{U_1}{1-\alpha}$$

we conclude that

$$\sum_{n=0}^{\infty} [\phi_{n+1}(i,\alpha,k) - \phi_n(i,\alpha,k)]$$

converges uniformly and that

$\phi_n(i,\alpha,k) \rightarrow \phi(i,\alpha,k)$

where $\phi(i,\alpha,k)$ has been previously defined.

Theorem 4: (1) of Chapter IV is optimal for the averaging cost criterion.

Proof: Let $(j_0^*, j_1^*, \ldots j_{k-1}^*, i_{k+1}^*, \ldots i_s^*)_\alpha = P_\alpha^*$

be the optimal policy for $\phi(i,\alpha,k)$. Let $\{\alpha_t\}$, with $\lim_{t\to\infty} \alpha_t = 1$ be a sequence such that $P_{\alpha_t} = P$ for all t.

Since the number of possible policies is finite such a sequence and, therefore, P* must exist. Let P be any other policy. Then certainly

$\phi_P(i,\alpha_t,k) \geq \phi(i,\alpha_t,k)$

Hence,

$$\lim_{t\to\infty} (1-\alpha_t)\ \phi_P(i,\alpha_t,k) \geq \lim_{t\to\infty} (1-\alpha_t)\ \phi(i,\alpha_t,k)$$

and again using the result that

$$\phi(i,k) = \lim_{t\to\infty} (1-\alpha_t)\ \phi(i,\alpha_t,k)$$

where $\alpha(i,k)$ has been previously defined, it is the minimum averaging cost.

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BIOGRAPHICAL SKETCH

Michael Jay Magazine was born in New York City on April 29, 1943. He was educated in New York elementary schools and graduated from Forest Hills High School in June, 1960. In June, 1964, he received the Bachelor of Science degree in Mathematics from the City College of New York. In January, 1966, he received the degree of Master of Science in Operations Research from New York University. While at New York University he worked as a graduate assistant in the Department of Industrial Engineering and Operations Research. Since September, 1966, Michael Magazine has been at the University of Florida pursuing the degree of Doctor of Philosophy. During this time he worked first as a research assistant and then as a research associate doing research and teaching courses in the area of Operations Research. In June, 1969, he was awarded the Master of Engineering degree from the University of Florida.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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