Generalizations of the Vitali-Hahn-Saks and Nikodym Theorems

By

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The Vitali-Hahn-Saks and Nikodym theorems are two very closely related results concerning sequences of countably additive, real valued measures. The purpose of this dissertation is to improve on these theorems both in the statements and the proofs. That is, stronger theorems will be shown to be true, and the proofs will be easier that those usually given. The author and J. K. Brooks extend the Vitali-Hahn-Saks and Nikodym theorems to the finitely additive vector case.

In 1969 J. K. Brooks used a result of Schur concerning uniform convergence of a double sequence of real numbers to derive simple, direct proofs of the Vitali-Hahn-Saks and Nikodym theorems. In Chapter 3 of this dissertation the above mentioned theorem of Schur is extended to the case where the double sequence is contained in a Banach space, and then this new result is used to obtain a short proof of the Nikodym theorem with the measures taking their values

in a Banach space.

However, the Vitali-Hahn-Saks and Nikodym theorems do not generalize directly to the case where the measures are finitely additive, bounded, and Banach valued, and the additional hypothesis of "strongly bounded" must be assumed. With this assumption, the theorems may be extended, and the proofs follow the same general line as those of Brooks' for the countably additive, scalar case. Instead of the Schur theorem, a generalized form of a result of Phillips is used, and the Nikodym theorem follows, but the proof of the Vitali-Hahn-Saks theorem requires a difficult construction with measurable sets. At the end of Chapter 4 a discussion is given to indicate how the theorems may be extended even further to the case where the measures take values in a locally convex linear topological space instead of a Banach space.

INTRODUCTION

This dissertation concerns the Vitali-Hahn-Saks and Nikodym theorems (7). In 1907 Vitali (21) proved the following theorem: if $f_n : [0,1] \to R$ are Lebesgue integrable functions converging almost everywhere to f, then $\lim_n \int_0^1 f_n ds$ and $\int_0^1 f ds$ exist and are equal if and only if the indefinite integrals of the f_n are uniformly absolutely continuous with respect to Lebesgue measure. Hahn (10) then proved in 1922 that if $f_n : [0,1] \to R$ are Lebesgue integrable functions and if $\lim_n \int_E f_n ds$ exists for every measurable set E, then the indefinite integrals of the f_n are uniformly absolutely continuous with respect to Lebesgue measure, and they converge to a set function which is also absolutely continuous with respect to Lebesgue measure. In 1933 Nikodym (12) generalized these two results when he proved that if (S, Σ, μ) is a measure space and if λ_n is a sequence of μ-continuous measures such that $\lim_n \lambda_n(E)$ exists for all E in Σ, then the λ_n are uniformly absolutely continuous with respect to μ. A little later Saks (19) gave another proof of Nikodym's theorem, and it became known as the Vitali-Hahn-Saks theorem. Using the countable additivity of the control measure, Saks defined a complete metric on the measurable sets, and then applied the Baire Category theorem. If the control measure had been only finitely additive, the metric would not necessarily have been complete, and this technique could not have been used.

Nikodym (12) proved tht if μ_n is a sequence of measures defined on a σ-algebra such that $\mu(E) = \lim_n \mu_n(E)$ exists for all E, then μ is countably additive and the countable additivity of the μ_n is uniform. This is known as the Nikodym theorem, and in most books it is proved as a consequence of the Vitali-Hahn-Saks theorem. Rickart (18) and Phillips (16) extended these theorems to the case where the measures are Banach space valued. The proofs of these results rely heavily on the fact that the set functions in question are countably additive, and there seems to be no way to generalize their techniques in order to prove Vitali-Hahn-Saks and Nikodym theorems for finitely additive, vector valued set functions.

However, in 1969 Brooks (3) gave a short proof of the Nikodym theorem and then used that result to prove the Vitali-Hahn-Saks theorem. His proof used a difficult construction with measurable sets along with a theorem of Schur (20) concerning the equivalence of weak and strong convergence in ℓ_1. In the same paper he gave a short proof that the truth of the two theorems in the scalar case would imply their truth in the vector case.

An extension of the Vitali-Hahn-Saks theorem to the finitely additive, scalar case has been proved by Ando (1), but his theorem was not extended to the case where the measures are vector valued. In 1970 Brooks and the author proved Vitali-Hahn-Saks and Nikodym theorems for finitely additive, strongly bounded, vector valued set functions, where strongly bounded is defined as in (17). The strongly bounded hypothesis is especially needed in the Nikodym theorem, because the statement would not even make sense if the measures were only bounded; the strongly bounded hypothesis may be dropped in the Vitali-Hahn-Saks theorem if the control measure is assumed to be finite. The proofs of the theorems are built around a generalization of a theorem of Phillips (15) and an extension of a technique of Darst (6).

The first two chapters provide the background material for the results proved later in the dissertation. However, Theorems 1.3 and 1.4 are stated only for completeness, and are not used in subsequent proofs. Although most of the theorems in these chapters may be found in Hilderbrandt (11) or Rickart (17), new proofs are given by the author.

The purpose of Chapter 3 is to generalize a theorem of Schur (20) and to use the new result to construct a direct proof of the vector case of the Nikodym theorem. Since all of the measures are assumed to be countably additive, the theorems rely only on the results of Chapter 1, and not of Chapter 2. Chapters 3 and 4 are independent, and no theorem of Chapter 3 is used in a proof in

Chapter 4.

The most important theorems are proved in Chapter 4. Both the Vitali-Hahn-Saks and Nikodym theorems are extended to the finitely additive case, and it is shown how the newer version of the Nikodym theorem implies the usual version when the measures are countably additive. The concept of "strongly bounded" is of utmost importance here, but Corollary 4.5 and the remark following the statement of Theorem 4.7 indicate when the condition may be dropped.

CHAPTER I

UNCONDITIONAL CONVERGENCE

Throughout this dissertation the following notation will be used.

Σ is a σ-algebra of subsets of a set S. $\mathcal{P}(\mathcal{N})$ is the power set of the natural numbers $\mathcal{N}$, and E and Δ are generic notations for sets belonging to Σ and $\mathcal{P}(\mathcal{N})$, respectively. R denotes the real numbers. $\mathcal{X}$ is a Banach space over the real or complex numbers, and $|x|$ is the norm of $x \in \mathcal{X}$. $\mathcal{X}^*$ is the conjugate space of $\mathcal{X}$, with the norm of an element in $\mathcal{X}^*$ defined in the usual way.

Let $n, m \in \mathcal{N}$ with $n \leq m$. Define $[n,m] = \{n, n+1, \cdots, m\}$, $[n,\infty) = \{n, n+1, \cdots\}$.

If μ is a real valued measure defined on a σ-algebra Σ, then $|\mu|$ is the total variation of μ, which is defined in the standard way.

In this chapter we define the term "unconditional convergence", and establish other conditions equivalent to it.

Definition: Let $x_n \in \mathfrak{X}$, $n = 1, 2, \cdots$. If $\pi : \mathcal{N} \to \mathcal{N}$ is a one-to-one, onto mapping, then $\{x_{\pi(n)}\}_{n=1}^{\infty}$ is called a rearrangement of $\{x_n\}_{n=1}^{\infty}$. The series $\sum_{n=1}^{\infty} x_n$ is said to be unconditionally convergent if the sum of every rearrangement of its terms converges.

Remark: A series that converges absolutely will converge unconditionally, but the converse is not necessarily true. In fact, Dvoretzky and Rogers (8) proved in 1950 that the two conditions are equivalent if and only if the Banach space in question is finite dimensional. The difference between absolute and unconditional convergence was further demonstrated by Brooks (4) in 1969 when he showed that weakly and strongly integrable functions with range in a Banach space correspond to unconditionally and absolutely convergent series in the space.

Theorem 1.1: Let $x_n \in \mathfrak{X}$, $n = 1, 2, \cdots$. The series $\sum_{n=1}^{\infty} x_n$

converges unconditionally if and only if every subsequential sum converges, that is, if $\{x_{n_i}\}$ is a subsequence, then $\sum_{i=1}^{\infty} x_{n_i}$ converges.

Proof: Let $\{x_{n_K}\}$ be a subsequence and assume the sum does not converge. Then there exist $\epsilon > 0$ and sequences $\{P_i\}$ and $\{V_i\}$ where for each i, $P_i < V_i < P_{i+1}$ and $\left|\sum_{K=P_i}^{V_i} x_{n_K}\right| > \epsilon$. Let $\{y_K\}_{K=1}^{\infty} = \{x_n\}_{n=1}^{\infty} - \bigcup_{i \in \eta} \{x_{n_K} : K = P_i, \ldots, V_i\}$. Then form the following rearrangement of $\{x_n\}$: $x_{n_{P_1}}, \ldots, x_{n_{V_1}}, y_1, x_{n_{P_2}}, \ldots, x_{n_{V_2}}, y_2, x_{n_{P_3}}, \ldots, x_{n_{V_3}}, y_3, x_{n_{P_4}}, \ldots, x_{n_{V_4}}, y_4, \ldots$.

The partial sums of this sequence are not Cauchy, so the sum does not exist; hence $\sum_{n=1}^{\infty} x_n$ does not converge unconditionally.

Conversely, assume $\{y_K\}$ is a rearrangement of $\{x_n\}$ and that the sum of $\{y_K\}$ does not exist. Write $y_P < y_V$ if y_P precedes y_V when considered in the ordering of the sequence $\{x_n\}$.

There exist $\epsilon > 0$ and sequences $\{m_i\}$ and $\{n_i\}$ where for all i, $m_i < n_i < m_{i+1}$ and $\left|\sum_{K=m_i}^{n_i} y_K\right| > \epsilon$. Let P_1 be such that for all $K \in [m_1, n_1]$ and all $t > P_1$, $y_K < y_t$. Rearrange $\{y_K : K = m_1, \ldots, n_1\}$ so that the elements appear in the same order as they appear in $\{x_n\}$, and call this rearrangement $\{x_{n_1}, \cdots, x_{n_{\omega_1}}\}$. Let $m_\ell > p_1$. Again rearrange $\{y_K : K = m_\ell, \cdots, n_\ell\}$ appropriately, and label this arrangement $\{x_{n_{\omega_1+1}}, \cdots, x_{n_{\omega_2}}\}$. Since $y_t < y_s$ if $t \in [m_1, n_1]$, $s \in [m_\ell, n_\ell]$, we have that $\{x_{n_1}, \cdots, x_{n_{\omega_1}}, x_{n_{\omega_1+1}}, \cdots x_{n_{\omega_2}}\}$ has the same order as $\{x_n\}$.

In this manner, define $\{n_K : K = 1, 2, \cdots\}$ and $\{\omega_i\}$ so that for every i , $\left| \sum_{K=\omega_i+1}^{\omega_{i+1}} x_{n_K} \right| > \epsilon$. Then $\{x_{n_K}\}$ is a subsequence whose sum does not exist; thus the converse is proved. Q.E.D.

<u>Theorem 1.2</u>: Let $x_n \in \mathfrak{X}$, $n = 1, 2, \cdots$. If $\sum_{n=1}^{\infty} x_n$ converges unconditionally, then for all $\epsilon > 0$, there exists an N such that for all $\Delta \subset [N, \infty)$, $\left| \sum_{n \in \Delta} x_n \right| < \epsilon$.

<u>Proof</u>: Assume the conclusion is false. Then there exists $\epsilon > 0$ such that for all N , there exists $\Delta \subset [N, \infty)$ such that $\left| \sum_{n \in \Delta} x_n \right| > \epsilon$.

Let $N_1 = 1$. Then there exists $\Delta_1 \subset [N_1, \infty)$ such that $\left| \sum_{n \in \Delta_1} x_n \right| > \epsilon$. Also, there exists Γ_1 , a finite subset of Δ_1 , such that $\left| \sum_{n \in \Gamma_1} x_n \right| > \frac{\epsilon}{2}$. Let $N_2 = \max \Gamma_1$. There exists

$\Delta_2 \subset [N_2+1,\infty)$ such that $\left|\sum_{n\in\Delta_2} x_n\right| \geq \epsilon$, so there exists a finite set $\Gamma_2 \subset \Delta_2$ such that $\left|\sum_{n\in\Gamma_2} x_n\right| > \frac{\epsilon}{2}$. Note that $\Gamma_1 \cap \Gamma_2 = \Box$, where $\Box$ denotes the null set. Also, for all $s \in \Gamma_1$, $t \in \Gamma_2$, $s < t$.

In a similar way define an infinite number of finite sets Γ_i such that for all i, $\left|\sum_{n\in\Gamma_i} x_n\right| > \frac{\epsilon}{2}$ and if $i < j$ and $s \in \Gamma_i$, $t \in \Gamma_j$ we have $s < t$.

$\{x_{n_\nu} : n_\nu \in \bigcup_{i=1}^{\infty} \Gamma_i\}$ is a subsequence of $\{x_n\}$. If m is any positive integer, there exists i_0 such that for all $n_\nu \in \Gamma_{i_0}$, $m < \nu$. Writing $\Gamma_{i_0} = \{n_{\nu_1}, \cdots, n_{\nu_2}\}$ we have that $\left|\sum_{\nu=\nu_1}^{\nu_2} x_{n_\nu}\right| = \left|\sum_{n\in\Gamma_{i_0}} x_n\right| > \frac{\epsilon}{2}$, so $\left\{\sum_{\nu=1}^{p} x_{n_\nu}\right\}_{p=1}^{\infty}$ is not a Cauchy sequence. Therefore the sum of the x_{n_ν} does not exist and this contradicts the unconditional convergence of $\sum_{n=1}^{\infty} x_n$.

<u>Theorem 1.3</u>: Let $x_n \in \mathfrak{X}$, $n = 1, 2, \cdots$. Then the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally if, and only if,

$\lim_{n} \sum_{m=n}^{\infty} |x^*(x_m)| = 0$ uniformly in x^*, where $|x^*| \leq 1$. The conclusion means that if $\epsilon > 0$, there exists an N so that for all x^* such that $|x^*| \leq 1$, we have $\sum_{m=n}^{\infty} |x^*(x_m)| < \epsilon$.

<u>Proof:</u> Assume the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally,

and let $\epsilon > 0$. By Theorem 1.2 there exists N so that $\left|\sum_{n\epsilon\Delta} x_n\right| < \frac{\epsilon}{2}$

for every $\Delta \subset [N,\infty)$. Let x^* be chosen so that $|x^*| \leq 1$ and define

$$\Delta_1 = \{n \epsilon [N,\infty) : x^*(x_n) \geq 0\}$$

$$\Delta_2 = \{n \epsilon [N,\infty) : x^*(x_n) < 0\} .$$

Then $\Delta_1 \cup \Delta_2 = [N,\infty)$, and $\Delta_1 \cap \Delta_2 = \Box$. Hence $\sum_{m=N}^{\infty} |x^*(x_m)| =$ $\sum_{m\epsilon\Delta_1} |x^*(x_m)| + \sum_{m\epsilon\Delta_2} |x^*(x_m)| = \left|\sum_{m\epsilon\Delta_1} x^*(x_m)\right| + \left|\sum_{m\epsilon\Delta_2} x^*(x_m)\right|$.

Note that for all $t > N$, $\left|\sum_{m\epsilon\Delta_1\cap[N,t]} x_m\right| < \frac{\epsilon}{2}$; therefore if $|x^*| \leq 1$, $\left|\sum_{m\epsilon\Delta_1\cap[N,t]} x^*(x_m)\right| = \left|x^*\sum_{m\epsilon\Delta_1\cap[N,t]} x_m\right| < \frac{\epsilon}{2}$. It follows that $\left|\sum_{m\epsilon\Delta_1} x^*(x_m)\right| = \left|\lim_t \sum_{m\epsilon\Delta_1\cap[N,t]} x^*(x_m)\right| = \left|\lim_t x^* \sum_{m\epsilon\Delta_1\cap[N,t]} x_m\right| \leq \frac{\epsilon}{2}$.

Likewise, it may be shown that $\left|\sum_{m\epsilon\Delta_2} x^*(x_m)\right| \leq \frac{\epsilon}{2}$; consequently $\sum_{m=N}^{\infty} |x^*(x_m)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ and the first implication of the theorem is proved.

Conversely, assume that the sum of the x_n does not converge unconditionally. Then by Theorem 1.1 there is a subsequence $\{x_{n_K}\}$ where the sum of the x_{n_K} does not exist; hence there is an $\epsilon > 0$ and sequences $\{P_i\}$ and $\{V_i\}$ where $P_i < V_i < P_{i+1}$ and for all i , $\left|\sum_{K=P_i}^{V_i} x_{n_K}\right| > \epsilon$. By the Hahn-Banach theorem there exists x_i^* for each i so that $|x_i^*| \leq 1$ and $\epsilon < \left|x_i^* \sum_{K=P_i}^{V_i} x_{n_K}\right| = \left|\sum_{K=P_i}^{V_i} x_i^*(x_{n_K})\right| \leq \sum_{K=P_i}^{V_i} |x_i^*(x_{n_K})|$.

For every N there is a value of i so that

$\{n_{p_i}, \cdots, n_{v_i}\} \subset [N, \infty)$; therefore $\sum_{n=N}^{\infty} |x_i^*(x_n)| \geq \sum_{K=p_i}^{v_i} |x_i^*(x_{n_K})| > \epsilon$.

Consequently the limit in the conclusion of the theorem does not converge uniformly in x^* where $|x^*| \leq 1$, and the second implication is proved. Q.E.D.

<u>Definition:</u> Let $x_n \epsilon \mathfrak{X}$, $n = 1, 2, \cdots$, and let $\mathfrak{S}$ be the collection of all finite sets σ contained in the natural numbers. Define a partial order on $\mathfrak{S}$ by $\sigma_1 \geq \sigma_2$ if $\sigma_1 \supseteq \sigma_2$. We say the series $\sum_{n=1}^{\infty} x_n$ <u>converges to a Moore-Smith limit</u> x_0 if for every $\epsilon > 0$, there exists σ_0 such that for all $\sigma \geq \sigma_0$, $\left| \sum_{n \epsilon \sigma} x_n - x_0 \right| < \epsilon$. Since $\mathfrak{X}$ is complete, an equivalent condition would be for every $\epsilon > 0$, there exists σ_0 so that if $\sigma_1, \sigma_2 \geq \sigma_0$, we have $\left| \sum_{n \epsilon \sigma_1} x_n - \sum_{n \epsilon \sigma_2} x_n \right| < \epsilon$.

<u>Theorem 1.4:</u> Let $x_n \epsilon \mathfrak{X}$, $n = 1, 2, \cdots$. The series $\sum_{n=1}^{\infty} x_n$ converges unconditionally if, and only if, it converges a Moore-Smith limit.

<u>Proof:</u> Assume $\sum_{n=1}^{\infty} x_n$ converges unconditionally. Let $\epsilon > 0$. By Theorem 1.2 there exists N so that for all $\Delta \subset [N, \infty)$,

$\left|\sum_{n\in\Delta} x_n\right| < \epsilon$. Let $\sigma_0 = [1, N]$. If $\sigma_1, \sigma_2 \geq \sigma_0$, we have that $\left|\sum_{n\in\sigma_1} x_n - \sum_{n\in\sigma_2} x_n\right| \leq \left|\sum_{n\in\sigma_1} x_n - \sum_{n\in\sigma_0} x_n\right| + \left|\sum_{n\in\sigma_2} x_n - \sum_{n\in\sigma_0} x_n\right| = \left|\sum_{n\in\sigma_1-\sigma_0} x_n\right| +$ $\left|\sum_{n\in\sigma_2-\sigma_0} x_n\right| < 2\epsilon$, because both $\sigma_1 - \sigma_0$ and $\sigma_2 - \sigma_0$ are contained in $[N,\infty)$. Therefore the series converges to a Moore-Smith limit.

Conversely, assume the series does not converge unconditionally, so there exists a subsequence $\{x_{n_K}\}$ whose sum does not exist. There exist $\epsilon > 0$ and sequences $\{P_i\}$, $\{V_i\}$ where $P_i < V_i < P_{i+1}$ and for each i , $\left|\sum_{K=P_i}^{V_i} x_{n_K}\right| > \epsilon$. Let σ_0 be any finite subset of the natural numbers, and choose i so that $\{n_{P_i}, \cdots, n_{V_i}\} \cap \sigma_0 = \square$. Letting $\sigma = \sigma_0 \cup \{n_{P_i}, \cdots, n_{V_i}\}$ we have that $\sigma \geq \sigma_0$ and $\left|\sum_{n\in\sigma} x_n - \sum_{n\in\sigma_0} x_n\right| > \epsilon$, and it follows that the series does not have a Moore-Smith limit. This completes the proof. Q.E.D.

CHAPTER 2

STRONGLY BOUNDED SET FUNCTIONS

Many theorems that are true for countably additive measures do not have extensions when the measures are finitely additive and bounded. However, as will be seen in Chapter 4, some of these results do carry over to the finitely additive case when the hypothesis of "strongly bounded" is assumed. In this chapter "strongly bounded" is defined and theorems are proved which will be used in Chapter 4.

Definition: Let $\mu : \Sigma \to \mathfrak{X}$ be a set function. We say μ is strongly bounded (s-bounded) if $\lim_K \mu(E_K) = 0$ whenever $\{E_K\}$ is a sequence of disjoint sets in Σ.

This definition was first given by Rickart (17).

Theorem 2.1. Let $\mu : \Sigma \to \mathfrak{X}$ be s-bounded and finitely additive. Then μ is bounded.

Proof: Assume μ is not bounded. Then there exists E such that $|\mu(E)| > 1$. Suppose there exists a monotone decreasing sequence of sets $F_K \subset E$ such that $|\mu(F_{K+1})| > |\mu(F_K)| + 1$ for all K. Then $\mu(F_K - F_{K+1}) - \mu(F_K) = -\mu(F_{K+1})$, and $|\mu(F_K - F_{K+1})| \geq |\mu(F_{K+1})| - |\mu(F_K)| > 1$. Therefore $\{F_K - F_{K+1}\}$ is a pairwise disjoint sequence of sets, whereas $\{\mu(F_K - F_{K+1})\}$ does not converge to zero, which contradicts the fact that μ is s-bounded.

Consequently there exists $G_1 \subset E$ such that $|\mu(G_1)| > 1$ and for all $L \subset G_1$, $|\mu(L)| \leq |\mu(G_1)| + 1$. There exists $R \varepsilon \Sigma$ such that $|\mu(R)| > |\mu(G_1)| + 2$, so $|\mu(R - G_1)| \geq |\mu(R)| - |\mu(R \cap G_1)| > 1$. If there exists a monotone decreasing sequence of sets $F_K \subset R - G_1$ such that $|\mu(F_{K+1})| > |\mu(F_K)| + 1$ for all K, we obtain a contradiction as before.

Therefore there exists $G_2 \subset R - G_1$ such that $|\mu(G_2)| > 1$ and for all $L \subset G_2$, $|\mu(L)| \leq |\mu(G_2)| + 1$. $G_1 \cap G_2 = \Box$. There exists P such that $|\mu(P)| > |\mu(G_1)| + |\mu(G_2)| + 3$, so $|\mu(P - (G_1 \cup G_2))| \geq |\mu(P)| - |\mu(P \cap G_1)| - |\mu(P \cap G_2)| > 1$, and there exists $G_3 \subset P - (G_1 \cup G_2)$

such that $|\mu(G_3)| > 1$ and for all $L \subset G_3$, $|\mu(L)| < |\mu(G_3)| + 1$. $G_3 \cap (G_1 \cup G_2) = \Box$.

Continue in this manner to define a sequence of disjoint sets $\{G_i\}$ such that for all i , $|\mu(G_i)| > 1$. This clearly contradicts the fact that μ is s-bounded, hence the theorem is proved. Q.E.D.

Remark: The converse to the above theorem is false. Let Σ be the Borel sets of the real line, and let $\mathfrak{X} = \mathcal{L}_\infty(R, \Sigma, \lambda)$, the class of essentially bounded measurable functions, where λ is Lebesgue measure (9). Define $\mu(E) = \xi_E$, the characteristic function of the set E . μ is bounded because for all E in Σ , $|\xi_E|_\infty \leq 1$. However, letting $E_n = (n, n+1)$, we see that $|\mu(E_n)|_\infty = 1$ for all n , so μ is not s-bounded.

On the other hand, if $\mathfrak{X}$ is the real numbers the converse holds, so the concepts of s-bounded and bounded coincide in the case of scalar measures.

Theorem 2.2: Let $\mu : \Sigma \to R$ be bounded and finitely additive. Then μ is s-bounded.

Proof: Assume the conclusion is false. Then there exists a sequence of disjoint sets $\{E_K\}$ where $\{\mu(E_K)\}_{K=1}^{\infty}$ does not converge to zero. Therefore there exists $\epsilon > 0$ and an infinite subcollection $\{E_{K_i}\} \subset \{E_K\}$ such that for all i, $|\mu(E_{K_i})| > \epsilon$. Let $\Delta \subset \mathcal{N}$ be such that for all $i \in \Delta$, $\mu(E_{K_i}) > \epsilon$ and for all $i \in \mathcal{N} - \Delta$, $\mu(E_{K_i}) < -\epsilon$. Either Δ or $\mathcal{N} - \Delta$ is infinite.

Without loss of generality, assume Δ is infinite. Let $M > 0$. Let Γ be a finite subset of Δ with cardinality greater than $\frac{M}{\epsilon}$. Then $\mu\left(\bigcup_{i\in\Gamma} E_{K_i}\right) = \sum_{i\in\Gamma} \mu(E_{K_i}) > \left(\frac{M}{\epsilon}\right)\epsilon = M$. Since M was chosen arbitrarily, this proves μ is not bounded - a contradiction.

Q.E.D.

Definition: Let $\mu : \Sigma \to \mathfrak{X}$. Define $\bar{\mu}(E) = \sup\{|\mu(E')| : E' \epsilon \Sigma, E' \subset E\}$. $\bar{\mu}$ is called the semi-variation of μ.

Theorem 2.3: Let $\mu : \Sigma \to \mathfrak{X}$. Then $\bar{\mu}$ is s-bounded if and only if μ is s-bounded.

Proof: Assume $\bar{\mu}$ is not s-bounded. Then there exists a disjoint sequence $\{E_K\}$ such that $\{\bar{\mu}(E_K)\}_{K=1}^{\infty}$ does not converge to zero. That is, there exists $\epsilon > 0$ and a subsequence $\{E_{K_i}\} \subset \{E_K\}$ such that for all i, $|\bar{\mu}(E_{K_i})| > \epsilon$. According to the definition of $\bar{\mu}$, for all i there exists $F_{K_i} \subset E_{K_i}$ such that $|\mu(F_{K_i})| > \frac{\epsilon}{2}$. Therefore $\{\mu(F_{K_i})\}_{i=1}^{\infty}$ does not converge to zero, and μ is not s-bounded. This proves that μ being s-bounded implies that $\bar{\mu}$ is s-bounded.

Conversely, since $|\mu(E)| \leq \bar{\mu}(E)$ for every E in Σ, $\lim_K \bar{\mu}(E_K) = 0$ implies that $\lim_K \mu(E_K) = 0$. Q.E.D.

<u>Theorem 2.4</u>: Let $\mu : \Sigma \to \mathcal{X}$ be finitely additive and s-bounded. If $\{E_K\}$ is a sequence of disjoint sets in Σ, then $\sum_{K=1}^{\infty} \mu(E_K)$ converges unconditionally.

<u>Proof</u>: Assume the conclusion is false. Then by Theorem 1.1 there exists a subsequence $\{E_{K_i}\} \subset \{E_K\}$ such that $\left\{\sum_{i=1}^{p} \mu(E_{K_i})\right\}_{p=1}^{\infty}$ does not converge and is therefore not Cauchy. Hence there exists $\epsilon > 0$ such that for all N there exists $p, q > N$ where $\left|\sum_{i=p}^{q} \mu(E_{K_i})\right| > \epsilon$.

Obtain p_1, q_1 such that $p_1 < q_1$ and $\left|\sum_{i=p_1}^{q_1} \mu(E_{K_i})\right| > \epsilon$, and then choose p_2 , q_2 such that $q_1 < p_2 < q_2$ and $\left|\sum_{i=p_2}^{q_2} \mu(E_{K_i})\right| > \epsilon$.

In this manner we define infinite collections $\{p_j\}$ and $\{q_j\}$ such that for all j , $p_j < q_j < p_{j+1}$, $\left|\sum_{i=p_j}^{q_j} \mu(E_{K_i})\right| > \epsilon$.

Let $F_j = \bigcup_{i=p_j}^{q_j} E_{K_i}$. Then $\{F_j\}$ is a sequence of disjoint sets and $|\mu(F_j)| = \left|\sum_{i=p_j}^{q_j} \mu(E_{K_i})\right| > \epsilon$, so $\{\mu(F_j)\}_{j=1}^{\infty}$ does not converge to zero and μ is not s-bounded. This proves the theorem.

Q.E.D.

CHAPTER 3

EXTENDING THE SCHUR THEOREM

In 1969 Brooks (3) proved the Nikodym theorem by using a result of Schur (20). In this chapter the Schur theorem is extended to the vector case, and the result is then used to obtain a very short proof of the Nikodym theorem for countably additive, Banach valued measures. The proof of the first theorem is a good example of the "sliding hump" technique.

<u>Theorem 3.1</u>: Let $x_{in} \in \mathcal{X}$, $i, n = 1, 2, \cdots$. Assume that for all i , the series $\sum_{n=1}^{\infty} x_{in}$ is unconditionally convergent, and also that:

$$(*) \quad \text{for all } \Delta \text{ , } \lim_{i} \sum_{n \in \Delta} x_{in} = 0 \text{ .}$$

Then the limit in (*) is uniform with respect to Δ . That is, for all ϵ , there exists I (depending on ϵ) such that for all $i \geq I$ and for all Δ , $\left| \sum_{n \in \Delta} x_{in} \right| < \epsilon$.

Remark: If $\mathfrak{X}$ is the real numbers, then $\sum_{n=1}^{\infty} x_{in}$ converges absolutely for every i, and the conclusion is that

$$\lim_i \sum_{n=1}^{\infty} |x_{in}| = 0 \quad .$$

Proof: Assume the limit in (*) does not exist uniformly with respect to Δ. Then there exist $\epsilon > 0$ and sequences $\{i_K\}$, $\{\Delta_K\}$ such that $\left|\sum_{n\in\Delta_K} x_{i_K n}\right| > \epsilon$. To simplify notation, let $i_K = K$, so $\left|\sum_{n\in\Delta_K} x_{Kn}\right| > \epsilon$ for all K. Let $K_1 = 1$. Since $\left|\sum_{n\in\Delta_1} x_{1n}\right| > \epsilon$, there exists a finite set $\Gamma_1 \subset \Delta_1$ such that $\left|\sum_{n\in\Gamma_1} x_{1n}\right| > \frac{2\epsilon}{5}$; due to the unconditional convergence of $\sum_{n=1}^{\infty} x_{1n}$, there exists R_1 such that for all $\Delta \subset [R_1, \infty)$, $\left|\sum_{n\in\Delta} x_{1n}\right| < \frac{\epsilon}{5}$. Let $T_1 = \max(\Gamma_1 \cup \{R_1\})$. If $\Delta \subset [T_1+1, \infty)$, we have $\Gamma_1 \cap \Delta = \Box$, and $\left|\sum_{n\in\Gamma_1\cup\Delta} x_{1n}\right| \geq \left|\sum_{n\in\Gamma_1} x_{1n}\right| - \left|\sum_{n\in\Delta} x_{1n}\right| > \frac{2\epsilon}{5} - \frac{\epsilon}{5} = \frac{\epsilon}{5}$.

For every n, $\lim_K x_{Kn} = 0$, so there exists I_n such that for all $i \geq I_n$, $|x_{in}| < \frac{\epsilon}{5T_1}$. Then $\sum_{n=1}^{T_1} |x_{in}| < \frac{\epsilon}{5}$ if $i \geq I = \max\{I_n : n \in [1, T_1]\}$. Let $K_2 = I$, then $\sum_{n\in\Delta_{K_2}} x_{K_2 n} = \sum_{n\in\Delta_{K_2}\cap[1,T_1]} x_{K_2 n} + \sum_{n\in\Delta_{K_2}-[1,T_1]} x_{K_2 n}$, and $\left|\sum_{n\in\Delta_{K_2}-[1,T_1]} x_{K_2 n}\right| \geq \left|\sum_{n\in\Delta_{K_2}} x_{K_2 n}\right| - \left|\sum_{n\in\Delta_{K_2}\cap[1,T_1]} x_{K_2 n}\right| > \epsilon - \sum_{n\in\Delta_{K_2}\cap[1,T_1]} |x_{K_2 n}| > \epsilon - \frac{\epsilon}{5} = \frac{4\epsilon}{5}$. Therefore there exists a finite set $\Gamma_2 \subset \Delta_{K_2} - [1, T_1]$ such that $\left|\sum_{n\in\Gamma_2} x_{K_2 n}\right| > \frac{3\epsilon}{5}$.

Since $\sum_{n=1}^{\infty} x_{K_2 n}$ converges unconditionally, there exists R_2 such that for all $\Delta \subset [R_2, \infty)$, $\left|\sum_{n\in\Delta} x_{K_2 n}\right| < \frac{\epsilon}{5}$. Let $T_2 = \max(\Gamma_2 \cup \{R_2\})$, so that for all $\Delta \subset [T_2+1, \infty)$ we have that $\Gamma_1 \cap \Delta = \Box$, $\Gamma_2 \cap \Delta = \Box$. Note that $\Gamma_1 \cap \Gamma_2 = \Box$, hence $\sum_{n\in\Gamma_2} x_{K_2 n} = \sum_{n\in\Gamma_1\cup\Gamma_2\cup\Delta} x_{K_2 n} - \sum_{n\in\Gamma_1} x_{K_2 n} - \sum_{n\in\Delta} x_{K_2 n}$, when $\Delta \subset [T_2+1, \infty)$. Also $\left|\sum_{n\in\Gamma_1} x_{K_2 n}\right| \le \sum_{n\in\Gamma_1} |x_{K_2 n}| < \frac{\epsilon}{5}$, because $\Gamma_1 \subset [1, T_1]$. Therefore

$$\left|\sum_{n\in\Gamma_1\cup\Gamma_2\cup\Delta} x_{K_2 n}\right| \ge \left|\sum_{n\in\Gamma_2} x_{K_2 n}\right| - \left|\sum_{n\in\Gamma_1} x_{K_2 n}\right| - \left|\sum_{n\in\Delta} x_{K_2 n}\right| > \frac{3\epsilon}{5} - \frac{\epsilon}{5} - \frac{\epsilon}{5} = \frac{\epsilon}{5} ,$$

for $\Delta \subset [T_2+1, \infty)$. Also, since $\Gamma_2 \cup \Delta \subset [T_1+1, \infty)$, we have that $\left|\sum_{n\in\Gamma_1\cup\Gamma_2\cup\Delta} x_{K_1 n}\right| > \frac{\epsilon}{5}$.

In the same manner as before, obtain I such that for all $i \ge I$, $\sum_{n=1}^{T_2} |x_{in}| < \frac{\epsilon}{5}$, and let $K_3 = I$. Then we have that $\left|\sum_{n\in\Delta_{K_3}-[1,T_2]} x_{K_3 n}\right| > \epsilon - \frac{\epsilon}{5} = \frac{4\epsilon}{5}$, and there exists a finite set $\Gamma_3 \subset \Delta_{K_3} - [1, T_2]$ such that $\left|\sum_{n\in\Gamma_3} x_{K_3 n}\right| > \frac{3\epsilon}{5}$. Choose R_3 such that for all $\Delta \subset [R_3, \infty)$, $\left|\sum_{n\in\Delta} x_{K_3 n}\right| < \frac{\epsilon}{5}$, and let $T_3 = \max(\Gamma_3 \cup \{R_3\})$. If $\Delta \subset [T_3+1, \infty)$, we then have $\left|\sum_{n\in\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Delta} x_{K_3 n}\right| > \frac{\epsilon}{5}$. Since $\Gamma_3 \cup \Delta \subset [T_2+1, \infty)$ we have that $\left|\sum_{n\in\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Delta} x_{K_2 n}\right| > \frac{\epsilon}{5}$, and $\Gamma_2 \cup \Gamma_3 \cup \Delta \subset [T_1+1, \infty)$ implies that $\left|\sum_{n\in\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Delta} x_{K_1 n}\right| > \frac{\epsilon}{5}$.

Continue to define an infinite number of Γ_j and K_j and let $\Gamma = \bigcup_{j=1}^{\infty} \Gamma_j$. Then for all j , $\left| \sum_{n \in \Gamma} x_{K_j n} \right| > \frac{\epsilon}{5}$ and $\left\{ \sum_{n \in \Gamma} x_{in} \right\}_{i=1}^{\infty}$ does not converge to zero, which contradicts the hypothesis.

Q.E.D.

Remark: Theorem 3.1 may be used to prove the equivalence of weak and strong convergence in ℓ_1 , the space of absolutely convergent series. Let $f_i, f \in \ell_1$, $i = 1, 2, \cdots$. Each element of ℓ_1 is a sequence of real numbers, and we write f_i as $\{x_{in}\}_{n=1}^{\infty}$ and f as $\{x_n\}_{n=1}^{\infty}$. The sequence $\{f_i\}$ converges strongly to f if $\lim_i \sum_{n=1}^{\infty} |x_{in} - x_n| = 0$, and $\{f_i\}$ converges weakly to f if $\{g^*(f_i)\}$ converges to $g^*(f)$ for every g^* in ℓ_1^*. We know that the conjugate space of ℓ_1 is ℓ_∞ , the space of all bounded sequences, and that if $g^* = \{y_n\}_{n=1}^{\infty}$, then

$$g^*(f_i) = \sum_{n=1}^{\infty} y_n x_{in} \quad .$$

Theorem 3.2: Let $f_i, f \in \ell_1$, $i = 1, 2, \cdots$. Then the f_i converge strongly to f if, and only if, they converge weakly to f .

Proof: Strong convergence always implies weak convergence, so I will prove the converse. Assume the f_i converge weakly to f. Writing f_i as $\{x_{in}\}_{n=1}^{\infty}$ and f as $\{x_n\}_{n=1}^{\infty}$, let $\alpha_{in} = x_{in} - x_n$. For any Δ, define y^* in ℓ_∞ by:

$$y^*(K) = \begin{cases} 1 & \text{if } K \in \Delta \\ 0 & \text{if } K \notin \Delta \end{cases}$$

Then $\lim_i \sum_{n\in\Delta} \alpha_{in} = \lim_i \sum_{n\in\Delta} (x_{in} - x_n) = \lim_i y^*(f_i - f) = 0$, because the f_i converge to f weakly. Therefore, by Theorem 3.1 we have that $\lim \sum_{n=1}^{\infty} |x_{in} - x_n| = 0$, and the f_i converge strongly to f. Note that we only needed to consider those elements of ℓ_∞ which take on the value 0 or 1. Q.E.D.

Theorem 3.3: Let $x_{in} \in \mathfrak{X}$, $i,n = 1,2,\cdots$. Assume that for all i, $\sum_{n=1}^{\infty} x_{in}$ converges unconditionally, and that:

(*) for all Δ, $\lim_i \sum_{n\in\Delta} x_{in}$ exists.

Then the limit in (*) is uniform with respect to Δ; that is, for every $\epsilon > 0$, there exists I such that for all $j \geq I$ and for all Δ, $\left| \sum_{n\in\Delta} x_{jn} - \lim_i \sum_{n\in\Delta} x_{in} \right| < \epsilon$.

Proof: This proof will be divided into six parts.

Part I: If $\{i_K\}$ is a subsequence, then

$\lim_{K} \sum_{n\in\Delta} (x_{i_K n} - x_{i_{K+1} n}) = 0$ uniformly with respect to Δ .

Proof of I: Let $\alpha_{Kn} = x_{i_K n} - x_{i_{K+1} n}$. Then for all K ,

$\sum_{n=1}^{\infty} \alpha_{Kn}$ converges unconditionally and for all Δ , $\lim_{K} \sum_{n\in\Delta} \alpha_{Kn} = 0$.

By Theorem 3.1, $\lim_{K} \sum_{n\in\Delta} \alpha_{Kn} = 0$ uniformly with respect to Δ ,

and the conclusion of I follows.

Part II: For all $\delta > 0$, there exists N such that for all integers i and for all $p \geq N$, $\left|\sum_{n=1}^{p} x_{in} - \sum_{n=1}^{\infty} x_{in}\right| < \delta$.

Proof of II: Suppose that we deny the conclusion. Then there exists $\delta > 0$ such that for all N there exist I and $p \geq N$ such that:

(1) $\left|\sum_{n=1}^{p} x_{In} - \sum_{n=1}^{\infty} x_{In}\right| \geq \delta$.

Choose δ as indicated in the above statement. Let N_0 be chosen arbitrarily. Then I claim that for all I , there exist $i > I$ and $p \geq N_0$ such that $\left|\sum_{n=1}^{p} x_{in} - \sum_{n=1}^{\infty} x_{in}\right| \geq \delta$. If this were not the case we could find I_0 such that for all $i > I_0$, $p \geq N_0$, $\left|\sum_{n=1}^{p} x_{in} - \sum_{n=1}^{\infty} x_{in}\right| < \delta$. For all i , $\sum_{n=1}^{\infty} x_{in}$ converges, so there exists N_i such that for all $p \geq N_i$, $\left|\sum_{n=1}^{p} x_{in} - \sum_{n=1}^{\infty} x_{in}\right| < \delta$.

Let $R = \max\{N_0, N_i : i=1,\cdots,I_0\}$. Then for all i and all $P \geq R$, $\left|\sum_{n=1}^{P} x_{in} - \sum_{n=1}^{\infty} x_{in}\right| < \delta$, and this contradicts the assumption (1). Therefore, since N_0 was picked arbitrarily, we have that for all N and I , there exist $i > I$, $P \geq N$ such that:

$$(2) \quad \left|\sum_{n=1}^{P} x_{in} - \sum_{n=1}^{\infty} x_{in}\right| \geq \delta .$$

Since $\lim_i \sum_{n=1}^{\infty} x_{in}$ exists, $\left\{\sum_{n=1}^{\infty} x_{in}\right\}_{i=1}^{\infty}$ is a Cauchy sequence, so there exists K such that for all $i,j \geq K$, $\left|\sum_{n=1}^{\infty} x_{in} - \sum_{n=1}^{\infty} x_{jn}\right| < \frac{\delta}{3}$. Let $i_1 = K$. Since $\sum_{n=1}^{\infty} x_{i_1 n}$ converges, we obtain N_1 such that for all $P \geq N_1$, $\left|\sum_{n=1}^{P} x_{i_1 n} - \sum_{n=1}^{\infty} x_{i_1 n}\right| < \frac{\delta}{3}$. According to (2), there exist $i_2 > I$, $v \geq N_1$ such that $\left|\sum_{n=1}^{v} x_{i_2 n} - \sum_{n=1}^{\infty} x_{i_2 n}\right| \geq \delta$. Therefore, since $\left(\sum_{n=1}^{v} x_{i_2 n} - \sum_{n=1}^{v} x_{i_1 n}\right) + \left(\sum_{n=1}^{v} x_{i_1 n} - \sum_{n=1}^{\infty} x_{i_1 n}\right) + \left(\sum_{n=1}^{\infty} x_{i_1 n} - \sum_{n=1}^{\infty} x_{i_2 n}\right) = \left(\sum_{n=1}^{v} x_{i_2 n} - \sum_{n=1}^{\infty} x_{i_2 n}\right)$, we have that $\left|\sum_{n=1}^{v} x_{i_2 n} - \sum_{n=1}^{v} x_{i_1 n}\right| \geq \left|\sum_{n=1}^{v} x_{i_2 n} - \sum_{n=1}^{\infty} x_{i_2 n}\right| - \left|\sum_{n=1}^{v} x_{i_1 n} - \sum_{n=1}^{\infty} x_{i_1 n}\right| - \left|\sum_{n=1}^{\infty} x_{i_1 n} - \sum_{n=1}^{\infty} x_{i_2 n}\right| > \delta - \frac{\delta}{3} - \frac{\delta}{3} = \frac{\delta}{3}$. Let $\Delta_1 = [1, v]$.

Now choose N_2 such that for all $s \geq N_2$, $\left|\sum_{n=1}^{s} x_{i_2 n} - \sum_{n=1}^{\infty} x_{i_2 n}\right| < \frac{\delta}{3}$. According to (2), obtain $i_3 > i_2$, $r \geq N_2$ such that $\left|\sum_{n=1}^{r} x_{i_3 n} - \sum_{n=1}^{\infty} x_{i_3 n}\right| \geq \delta$, and since i_3 and i_2 are both greater than K , we also have that $\left|\sum_{n=1}^{\infty} x_{i_3 n} - \sum_{n=1}^{\infty} x_{i_2 n}\right| < \frac{\delta}{3}$. Combining these inequalities, we obtain $\left|\sum_{n=1}^{r} x_{i_2 n} - \sum_{n=1}^{r} x_{i_3 n}\right| > \frac{\delta}{3}$. Let $\Delta_2 = [1, r]$.

Continuing this process, define $\{i_m\}$ and $\{\Delta_m\}$, $m = 1, 2, \cdots$ such that $\left|\sum_{n \in \Delta_m} x_{i_m n} - \sum_{n \in \Delta_m} x_{i_{m+1} n}\right| > \frac{\delta}{3}$ for all m . Then $\lim \sum_{n \in \Delta} (x_{i_m n} - x_{i_{m+1} n})$ does not converge to zero uniformly with respect to Δ , which contradicts Part I; hence, the conclusion of Part II is proved.

Part III: If $s \in \mathcal{N}$, then $\lim_i \sum_{n=1}^{s} x_{in} = \sum_{n=1}^{s} x_n$, where $x_n = \lim_i x_{in}$.

Proof of III: This follows because the sums in question are finite, hence the limit may be interchanged.

Part IV: For all Δ , $\sum_{n \in \Delta} x_n$ exists.

Proof of IV: If Δ is finite, the conclusion holds, so assume Δ is infinite. Let $\Delta = \{n_K\}$. Letting $\beta_{iK} = x_{i n_K}$ we see that $\{\beta_{iK}\}$ satisfies the hypothesis of Theorem 3.3; consequently, we may apply the results of Parts I, II, and III to the double sequence $\{\beta_{iK}\}$. To show $\sum_{n \in \Delta} x_n$ exists it suffices to show that $\sum_{K=1}^{\infty} \beta_K$ exists, where $\beta_K = \lim_i \beta_{iK}$.

In order to prove $\sum_{K=1}^{\infty} \beta_K$ exists, I will show that $\left\{\sum_{K=1}^{V} \beta_K\right\}_{V=1}^{\infty}$ is a Cauchy sequence. Let $\epsilon > 0$. By Part II there exists an N such that for every i and $V \geq N$, $\left|\sum_{K=1}^{V} \beta_{iK} - \sum_{K=1}^{\infty} \beta_{iK}\right| < \frac{\epsilon}{6}$. Therefore, if $V, S \geq N$ we have $\left|\sum_{K=1}^{V} \beta_{iK} - \sum_{K=1}^{S} \beta_{iK}\right| < \frac{\epsilon}{3}$ for all i . From Part III choose I such that for all $i \geq I$, $\left|\sum_{K=1}^{V} \beta_{iK} - \sum_{K=1}^{V} \beta_K\right| < \frac{\epsilon}{3}$ and $\left|\sum_{K=1}^{S} \beta_{iK} - \sum_{K=1}^{S} \beta_K\right| < \frac{\epsilon}{3}$. Then for all $i \geq I$,

$$\left|\sum_{K=1}^{V} \beta_K - \sum_{K=1}^{S} \beta_K\right| \leq \left|\sum_{K=1}^{S} \beta_K - \sum_{K=1}^{S} \beta_{iK}\right| + \left|\sum_{K=1}^{S} \beta_{iK} - \sum_{K=1}^{V} \beta_{iK}\right| + \left|\sum_{K=1}^{V} \beta_{iK} - \sum_{K=1}^{V} \beta_K\right| < \epsilon .$$

Since ϵ was chosen arbitrarily, it follows that $\left\{\sum_{K=1}^{V} \beta_K\right\}_{V=1}^{\infty}$ is Cauchy, and the conclusion of Part IV is proved.

For all Δ , $\sum_{n\epsilon\Delta} x_n$ exists, and Theorem 1.1 indicates that $\sum_{n=1}^{\infty} x_n$ converges unconditionally.

Part V: For all Δ , $\lim \sum_{n\epsilon\Delta} x_{in} = \sum_{n\epsilon\Delta} x_n$. (Note that the second sum is defined by the result of Part IV.)

Proof of V: Since the conclusion is true for finite Δ , we may assume Δ is infinite.

Let $\Delta = \{n_K\}$, $\beta_{iK} = x_{in_K}$. We need to show that $\lim_i \sum_{K=1}^{\infty} \beta_{iK} = \sum_{K=1}^{\infty} \beta_K$, and since $\{\beta_{iK}\}$ satisfies the hypothesis of Theorem 3.3, we may apply the results of the first four parts.

Let $\epsilon > 0$. According to Parts II and IV, choose N so that $\left|\sum_{K=1}^{N}\beta_K - \sum_{K=1}^{\infty}\beta_K\right| < \epsilon$ and for every i , $\left|\sum_{K=1}^{N}\beta_{iK} - \sum_{K=1}^{\infty}\beta_{iK}\right| < \epsilon$.

By Part III, choose I so that for every $i \geq I$,

$\left|\sum_{K=1}^{N}\beta_{iK} - \sum_{K=1}^{N}\beta_K\right| < \epsilon$. Then for every $i \geq I$, $\left|\sum_{K=1}^{\infty}\beta_{iK} - \sum_{K=1}^{\infty}\beta_K\right| \leq$

$\left|\sum_{K=1}^{\infty}\beta_{iK} - \sum_{K=1}^{N}\beta_{iK}\right| + \left|\sum_{K=1}^{N}\beta_{iK} - \sum_{K=1}^{N}\beta_K\right| + \left|\sum_{K=1}^{\infty}\beta_K - \sum_{K=1}^{N}\beta_K\right| < 3\epsilon$.

Since ϵ was chosen arbitrarily, $\lim_i \sum_{K=1}^{\infty}\beta_{iK} = \sum_{K=1}^{\infty}\beta_K$.

Part VI: The proof of the conclusion of the theorem.

Proof of VI: Let $\alpha_{in} = x_{in} - x_n$. By Part IV, $\sum_{n=1}^{\infty}x_n$ converges unconditionally, so for all i , $\sum_{n=1}^{\infty}\alpha_{in}$ converges unconditionally. Also, for all Δ , $\lim_i \sum_{n\in\Delta}\alpha_{in} = \lim_i \sum_{n\in\Delta}(x_{in} - x_n) =$ $\lim_i \sum_{n\in\Delta}x_{in} - \sum_{n\in\Delta}x_n = 0$, where the last inequality results from Part V.

Therefore, by Theorem 3.1, for all $\epsilon > 0$, there exists I such that for all $i \geq I$ and all Δ , $\epsilon > \left|\sum_{n\in\Delta}\alpha_{in}\right| =$ $\left|\sum_{n\in\Delta}x_{in} - \sum_{n\in\Delta}x_n\right| = \left|\sum_{n\in\Delta}x_{in} - \lim_i \sum_{n\in\Delta}x_{in}\right|$, where the last equality follows from Part V. This concludes the proof of the theorem.

Q.E.D.

Theorem 3.3 is the desired extension of the Schur theorem, and will be used to prove Theorem 3.4, the Nikodym theorem.

<u>Definition</u>: Let $\mu : \Sigma \to \mathfrak{X}$. μ is <u>countably</u> <u>additive</u> if $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$ whenever $\{E_n\}$ is a sequence of disjoint sets in Σ .

<u>Remark</u>: μ being countably additive implies μ is s-bounded because if μ is not s-bounded, then there exist $\epsilon > 0$ and disjoint sets $\{E_n\}$ such that for all n , $|\mu(E_n)| > \epsilon$; hence $\left\{\sum_{n=1}^{K} \mu(E_n)\right\}_{K=1}^{\infty}$ is not a Cauchy sequence and μ is not countably additive.

<u>Definition</u>: Let $\mu_n : \Sigma \to \mathfrak{X}$ be countably additive, $n = 1, 2, \cdots$, and let $\{E_i\}$ be a sequence of disjoint sets in Σ . The μ_n are <u>uniformly</u> <u>countably</u> <u>additive</u> if for all $\epsilon > 0$ there exists M such that for all $m \geq M$ and for all n , $\left|\sum_{i=1}^{m} \mu_n(E_i) - \sum_{i=1}^{\infty} \mu_n(E_i)\right| < \epsilon$.

<u>Theorem 3.4 (Nikodym)</u>: Let $\mu_n : \Sigma \to \mathfrak{X}$ be countably additive, $n = 1, 2, \cdots$. If $\mu(E) = \lim_n \mu_n(E)$ exists for every E , then μ is countably additive and the countable additivity of the μ_n is uniform.

<u>Proof</u>: Let $\{E_n\}$ be a pairwise disjoint sequence of sets

in Σ and let $x_{in} = \mu_i(E_n)$. The double sequence $\{x_{in}\}$ satisfies the hypothesis of Theorem 3.3, so by the result of Part III of the proof of that theorem, for every $\delta > 0$, there exists N such that for every integer i and all $P \geq N$, $\delta > \left|\sum_{n=1}^{P} x_{in} - \sum_{n=1}^{\infty} x_{in}\right| = \left|\sum_{n=1}^{P} \mu_i(E_n) - \sum_{n=1}^{\infty} \mu_i(E_n)\right|$. Therefore the μ_i are uniformly countably additive.

On the other hand, $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_i \mu_i\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_i \sum_{n=1}^{\infty} \mu_i(E_n) = \lim_i \sum_{n=1}^{\infty} x_{in} = \sum_{n=1}^{\infty} x_n$, the last equality following from Part V of the proof of Theorem 3.3. Since $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \mu(E_n)$, we have that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$, and μ is countably additive. Q.E.D.

We give a proof due to Brooks and Mikusinski (5) of a result of Banach (2).

Lemma 3.5: Let $\mathfrak{X}$ be a separable Banach space, and let $\{y_n^*\}$ be a bounded sequence in $\mathfrak{X}^*$. Then there exists a subsequence $\{y_{n_K}^*\}$ such that $\lim_K y_{n_K}^*(x)$ exists for every x in $\mathfrak{X}$.

Proof: This proof consists of a standard Cantor diagonal process. Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of $\mathfrak{X}$, and choose $M > 0$ such that $|y_n^*| < M$ for every n. We have that $|y_n^*(x_1)| \leq M|x_1|$

for each n, so $\{y^*_n(x_1)\}$ is a bounded sequence of real numbers, and there exists a subsequence $\{y^*_{1,K}\}$ of $\{y^*_n\}$ so that $\lim_K y^*_{1,K}(x_1)$ exists. Likewise, there exists a subsequence $\{y^*_{2,K}\}$ of $\{y^*_{1,K}\}$ such that $\lim_K y^*_{2,K}(x_2)$ exists. Continue in this manner to define subsequences $\{y^*_{n,K}\}$ so that $\{y^*_{n+1,K}\}_{K=1}^{\infty} \subset \{y^*_{n,K}\}_{K=1}^{\infty}$ and $\lim_K y^*_{n,K}(x_n)$ exists for every n. For each integer K, let $y^*_{n_K} = y^*_{K,K}$. Then $\{y^*_{n_K}\}$ is a subsequence of $\{y^*_n\}$, and for all j, $\lim_K y^*_{n_K}(x_j)$ exists.

We now show that $\lim_K y^*_{n_K}(x)$ exists for each $x \in \mathfrak{X}$. Let $x \in \mathfrak{X}$, and $\epsilon > 0$. Since $\{x_j\}_{j=1}^{\infty}$ is dense in $\mathfrak{X}$, pick x_j so that $|x_j - x| < \frac{\epsilon}{3M}$, and choose R such that for all $K, \ell \geq R$, $|(y^*_{n_K} - y^*_{n_\ell})(x_j)| < \frac{\epsilon}{3}$. Then for all $K, \ell \geq R$,

$$|y^*_{n_K}(x) - y^*_{n_\ell}(x)| \leq |y^*_{n_K}(x) - y^*_{n_K}(x_j)| + |y^*_{n_K}(x_j) - y^*_{n_\ell}(x_j)| + |y^*_{n_\ell}(x_j) - y^*_{n_\ell}(x)| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon .$$

Therefore the sequence $\{y^*_{n_K}(x)\}$ is Cauchy, and $\lim_K y^*_{n_K}(x)$ exists. Q.E.D.

<u>Definition</u>: Let $x_n \in \mathfrak{X}$, $n = 1, 2, \cdots$. $\sum_{n=1}^{\infty} x_n$ is <u>weakly unconditionally convergent</u> if for every subset Δ of the integers, there exists x_Δ such that $\sum_{n \in \Delta} x^*(x_n) = x^*(x_\Delta)$ for all x^* in $\mathfrak{X}^*$.

The following theorem is known as the Orlicz-Pettis theorem, (13, 14), and the proof is due to Brooks and Mikusinski (5).

Theorem 3.6: If $\mathfrak{X}$ is a separable Banach space, then a series $\sum_{n=1}^{\infty} x_n$ converges unconditionally if, and only if, it converges weakly unconditionally.

Proof: Unconditional convergence always implies weak unconditional convergence. In order to prove the converse, I will assume that the series converges weakly unconditionally but not unconditionally, and then arrive at a contradiction.

If $\sum_{n=1}^{\infty} x_n$ does not converge unconditionally, there exist an $\epsilon > 0$ and finite disjoint sets Δ_i such that $\left|\sum_{n \in \Delta_i} x_n\right| > \epsilon$ for every integer i. Letting $z_i = \sum_{n \in \Delta_i} x_n$, we have that $\sum_{i=1}^{\infty} z_i$ also converges weakly unconditionally.

According to the Hahn-Banach theorem, there exist x_m^* such that $|x_m^*| = 1$ and $|x_m^*(z_m)| = |z_m|$ for every integer m. By Lemma 3.5 there is a subsequence $\{x_{m_i}^*\}$ of $\{x_m^*\}_{m=1}^{\infty}$ such that $\lim_i x_{m_i}^*(x)$ exists for every x in $\mathfrak{X}$, and to simplify notation I will assume that $m_i = i$ for all i. Let $\alpha_{in} = x_i^*(z_n)$.

For any Δ, there exists z_Δ so that $\sum_{n\in\Delta} x_i^*(z_n) = x_i^*(z_\Delta)$ for all i. Therefore, $\lim_i \sum_{n\in\Delta} \alpha_{in} = \lim_i \sum_{n\in\Delta} x_i^*(z_n) = \lim_i x_i^*(z_\Delta)$. Since $\sum_{n\in\Delta} x_i^*(z_n)$ exists for all Δ and unconditional convergence implies absolute convergence in the scalar field, it follows that $\sum_{n=1}^{\infty} |x_i^*(z_n)| < \infty$. Therefore, the double sequence $\{\alpha_{in}\}$ satisfies the hypothesis of Theorem 3.3.

Letting $\alpha_n = \lim_i \alpha_{in}$, we have from Part IV of the proof of Theorem 3.3 that $\sum_{n=1}^{\infty} |\alpha_n| < \infty$, and from the conclusion of the theorem we have that $\lim_i \left| \sum_{n\in\Delta} \alpha_{in} - \sum_{n\in\Delta} \alpha_n \right| = 0$ uniformly in Δ. Choose positive integers N_1 and N_2 such that for all $n \geq N_1$, $|\alpha_n| < \frac{\epsilon}{3}$, and for all $i \geq N_2$ and all n, $|\alpha_{in} - \alpha_n| < \frac{\epsilon}{3}$. Let $m > \max\{N_1, N_2\}$. Then $\epsilon < |z_m| = |x_m^*(z_m)| = |\alpha_{mm}| \leq |\alpha_{mm} - \alpha_m| + |\alpha_m| < \frac{2\epsilon}{3}$, which is a contradiction. Hence the theorem is proved. Q.E.D.

CHAPTER 4

EXTENSIONS OF THE VITALI-HAHN-SAKS AND NIKODYM THEOREMS

In this chapter the Vitali-Hahn-Saks and Nikodym theorems are extended to the case where the measures are finitely additive and strongly bounded. The first theorem to be proved is the key to the whole chapter, and is an extension of a result of Phillips (15).

Theorem 4.1: Let $\mu_n: \mathcal{P}(\mathcal{N}) \to \mathfrak{X}$ be finitely additive and s-bounded. If $\lim_n \mu_n(\Delta) = 0$ for all Δ, then $\lim_n \sum_{t\in\Delta} \mu_n(t) = 0$ uniformly in Δ. (That is, for every $\epsilon > 0$ there exists N such that for all $n \geq N$ and for all Δ, $\left|\sum_{t\in\Delta} \mu_n(t)\right| < \epsilon$.)

Remark: Since the μ_n are finitely additive and s-bounded, it follows from Theorem 2.2 that $\sum_{t=1}^{\infty} \mu_n(t)$ converges unconditionally. Therefore $\sum_{t\in\Delta} \mu_n(t)$ exists for every Δ and the conclusion makes sense.

Proof: Assume the conclusion is false. Then there exist an $\epsilon > 0$ and sequences $\{n_i\}$, $\{\Delta_i\}$ such that $\left|\sum_{t\in\Delta_i} \mu_{n_i}(t)\right| > \epsilon$, $i = 1, 2, \ldots$. To simplify notation, assume $n_i = i$.

Let $i_1 = 1$. $\left|\sum_{t\in\Delta_1} \mu_1(t)\right| > \epsilon$, so there exists N_1 such that $\left|\sum_{t\in\Delta} \mu_1(t)\right| < \frac{\epsilon}{16}$ for all $\Delta \subset [N_1, \infty)$. Letting $\tau_1 = \Delta_1 \cap [1, N_1 - 1]$ we have $\left|\sum_{t\in\tau_1} \mu_1(t)\right| = \left|\sum_{t\in\Delta_1\cap[1,N_1-1]} \mu_1(t)\right| \geq \left|\sum_{t\in\Delta_1} \mu_1(t)\right| - \left|\sum_{t\in\Delta_1\cap[N_1,\infty)} \mu_1(t)\right| > \frac{\epsilon}{2}$.

Since $\lim_n \mu_n(t) = 0$ for all t, we can pick i_2 such that $\sum_{t=1}^{N_1-1} |\mu_{i_2}(t)| < \frac{\epsilon}{16}$. $\sum_{t=1}^{\infty} \mu_{i_2}(t)$ converges unconditionally, so we can pick $N_2 > N_1$ such that for all $\Delta \subset [N_2, \infty)$, $\left|\sum_{t\in\Delta} \mu_{i_2}(t)\right| < \frac{\epsilon}{16}$. As a result, $\sum_{t\in\Delta_{i_2}} \mu_{i_2}(t) = \sum_{t\in\Delta_{i_2}\cap[1,N_1-1]} \mu_{i_2}(t) + \sum_{t\in\Delta_{i_2}\cap[N_1,N_2-1]} \mu_{i_2}(t) + \sum_{t\in\Delta_{i_2}\cap[N_2,\infty)} \mu_{i_2}(t)$, and therefore $\left|\sum_{t\in\Delta_{i_2}\cap[N_1,N_2-1]} \mu_{i_2}(t)\right| \geq \left|\sum_{t\in\Delta_{i_2}} \mu_{i_2}(t)\right| - \left|\sum_{t\in\Delta_{i_2}\cap[N_2,\infty)} \mu_{i_2}(t)\right| - \sum_{t\in\Delta_{i_2}\cap[1,N_1-1]} |\mu_{i_2}(t)| > \epsilon - \frac{\epsilon}{16} - \frac{\epsilon}{16} > \frac{\epsilon}{2}$. Let $\tau_2 = \Delta_{i_2} \cap [N_1, N_2 - 1]$.

Choose i_3 such that $\sum_{t=1}^{N_2-1} |\mu_{i_3}(t)| < \frac{\epsilon}{16}$ and pick N_3 so that for all Δ, $\left|\sum_{t\in\Delta\cap[N_3,\infty)} \mu_{i_3}(t)\right| < \frac{\epsilon}{16}$. Then, as before,

$\left|\sum_{t\epsilon\Delta_{i_3}\cap[N_2,N_3-1]}\mu_{i_3}(t)\right| > \frac{\epsilon}{2}$, and we let $\tau_3 = \Delta_{i_3}\cap[N_2,N_3-1]$.

Continue in this manner to define an increasing sequence of positive integers $\{N_i\}$ such that:

(1) $\sum_{t=1}^{N_{K-1}-1}|\mu_{i_K}(t)| < \frac{\epsilon}{16}$, $K=2,3,\cdots$;

(2) $\left|\sum_{t\epsilon\Delta\cap[N_K,\infty)}\mu_{i_K}(t)\right| < \frac{\epsilon}{16}$ for all Δ and $K=2,3,\cdots$;

(3) letting $\tau_K = \Delta_K\cap[N_{K-1},N_K-1]$, $(N_0=1)$, we have $|\mu_{i_K}(\tau_K)| > \frac{\epsilon}{2}$.

<u>Note</u>: If Δ is a finite set disjoint from $[N_{K-1},N_K-1]$, then it follows from (1) and (2) that $|\mu_{i_K}(\Delta)| < \frac{\epsilon}{8}$.

Let $B_m=\{\tau_{2^n(2m-1)}\}_{n=1}^{\infty}$, $m=1,2,\cdots$. For all values of m , B_m is an infinite set. Also the sets $\{B_m\}$ are disjoint because $m_1\neq m_2$ implies that $2^{n_1}(2m_1-1)\neq 2^{n_2}(2m_2-1)$ for all values of n_1 and n_2 . Since the sequence $\{\tau_K\}$ is disjoint, it then follows from the preceding statement that $\{\cup B_m\}_{m=1}^{\infty}$ is a sequence of disjoint sets, where $\cup B_m$ is defined to be $\cup\{\tau:\tau\epsilon B_m\}$. In other words, I have divided a countably infinite collection of sets into a countable number of countably infinite collections; and then have taken the union of each collection. Since all the sets were disjoint, all the unions are disjoint from each other.

Since μ_{i_1} is s-bounded it follows from Theorem 2.3 that $\bar{\mu}_{i_1}$ is s-bounded, so $\lim_m \bar{\mu}_{i_1}(\cup B_m)=0$. Hence there exists an m_0 so that $\bar{\mu}_{i_1}(\cup B_{m_0})<\frac{\epsilon}{8}$. Therefore for all $\Delta \subset \cup B_{m_0}$, $|\mu_{i_1}(\Delta)|<\frac{\epsilon}{8}$. $\tau \notin B_{m_0}$ because $|\mu_{i_1}(\tau)|>\frac{\epsilon}{2}$.

Let $\Gamma_1 = B_{m_0}$. Γ_1 is a countable collection of the τ_K, so by the same process as above there exists Γ_2, an infinite subset of Γ_1, such that $\bar{\mu}_{i_2}(\cup\Gamma_2)<\frac{\epsilon}{8}$. Again $\tau_2 \notin \Gamma_2$. Continue this process to obtain a sequence of infinite sets $\{\Gamma_n\}$ where $\Gamma_{n+1} \subset \Gamma_n$ and $|\bar{\mu}_{i_n}(\cup\Gamma_n)|<\frac{\epsilon}{8}$.

Let Γ consist of the first element of Γ_n for every n. Γ is an infinite subset of $\{\tau_K\}$ and for each value of n, $\Gamma-\Gamma_n$ consists of a finite number of elements.

If $\tau_K \epsilon \Gamma$ then $|\mu_{i_K}(\cup\Gamma)| \geq |\mu_{i_K}(\tau_K)| - |\mu_{i_K}(\cup[\Gamma-(\Gamma_K \cup \{\tau_K\})])| - |\mu_{i_K}(\cup[\Gamma \cap \Gamma_K])|$, because $\tau_K \notin \Gamma_K$.

$\cup[\Gamma-(\Gamma_K \cup \{\tau_K\})]$ is a finite set disjoint from $[N_{K-1}, N_K - 1]$, so $|\mu_{i_K}(\cup[\Gamma-(\Gamma_K\cup\{\tau_K\})])|<\frac{\epsilon}{8}$. $\cup(\Gamma\cap\Gamma_K) \subset \cup\Gamma_K$, so

$|\mu_{i_K}(\cup(\Gamma \cap \Gamma_K))| < \frac{\epsilon}{8}$. Combining all three inequalities we obtain $|\mu_{i_K}(\cup\Gamma)| \geq |\mu_{i_K}(\tau_K)| - \frac{\epsilon}{8} - \frac{\epsilon}{8} > \frac{\epsilon}{2} - \frac{\epsilon}{8} - \frac{\epsilon}{8} = \frac{\epsilon}{4}$.

Therefore $|\mu_{i_K}(\cup\Gamma)| > \frac{\epsilon}{4}$ if $\tau_K \in \Gamma$. There are an infinite number of τ_K in Γ , so $\{\mu_{i_K}(\cup\Gamma)\}_{K=1}^{\infty}$ cannot converge to zero, which contradicts the hypothesis. This proves the theorem. Q.E.D.

Remark: Theorem 4.1 implies Theorem 3.1.

Corollary 4.2: Let $\mu_n : \Sigma \to \mathfrak{X}$ be finitely additive and s-bounded, $n = 1, 2, \cdots$. Assume $\lim_n \mu_n(E)$ exists for all E . Then if $\{E_K\}$ is a sequence of disjoint sets, $\lim_n \sum_{K \in \Delta} (\mu_{n+1} - \mu_n)(E_K) = 0$ uniformly in Δ .

Proof: Define $\nu_n : \mathcal{P}(\mathcal{N}) \to \mathfrak{X}$ as follows: $\nu_n(\Delta) = (\mu_{n+1} - \mu_n)(\bigcup_{K \in \Delta} E_K)$. The ν_n satisfy the hypothesis of Theorem 4.1, so $\lim_n \sum_{K \in \Delta} (\mu_{n+1} - \mu_n)(E_K) = \lim_n \sum_{K \in \Delta} \nu_n(K) = 0$ uniformly in Δ . Q.E.D.

Corollary 4.3: Let $\mu_n : \Sigma \to \mathfrak{X}$ be finitely additive and s-bounded, $n = 1, 2, \cdots$. Assume $\lim_n \mu_n(E)$ exists for all E , and let $\mu(E) = \lim_n \mu_n(E)$. Then μ is s-bounded.

Proof: Assume μ is not s-bounded. Then there exist $\epsilon > 0$ and a sequence of disjoint sets $\{E_K\}$ such that for all K, $|\mu(E_K)| > \epsilon$. Let $n_1 = 1$. Since $\lim_K \mu_{n_1}(E_K) = 0$, there exists K_1 such that $|\mu_{n_1}(E_{K_1})| < \frac{\epsilon}{3}$. However $\lim_n |\mu_n(E_{K_1})| = |\mu(E_{K_1})| > \epsilon$ so there exists $n_2 > n_1$ so that $|\mu_{n_2}(E_{K_1})| > \frac{2\epsilon}{3}$. Therefore $|(\mu_{n_2} - \mu_{n_1})(E_{K_1})| > \frac{2\epsilon}{3} - \frac{\epsilon}{3} = \frac{\epsilon}{3}$. Because $\lim_K \mu_{n_2}(E_K) = 0$, there exists K_2 such that $|\mu_{n_2}(E_{K_2})| < \frac{\epsilon}{3}$. $\lim_n |\mu_n(E_{K_2})| = |\mu(E_{K_2})| > \epsilon$, so there exists $n_3 > n_2$ so that $|\mu_{n_3}(E_{K_2})| > \frac{2\epsilon}{3}$ and $|(\mu_{n_2} - \mu_{n_3})(E_{K_2})| > \frac{2\epsilon}{3} - \frac{\epsilon}{3} = \frac{\epsilon}{3}$.

Continue this process to define sequences $\{n_i\}$, $\{K_j\}$ where $\{n_i\}$ is monotone increasing and for every integer i , $|(\mu_{n_{i+1}} - \mu_{n_i})(E_{K_i})| > \frac{\epsilon}{3}$. Therefore the sequence $\left\{\sum_{j \in \Delta} (\mu_{n_{i+1}} - \mu_{n_i})(E_{K_j})\right\}_{i=1}^{\infty}$ does not converge to zero uniformly in Δ , and this contradicts Corollary 4.2. Hence the theorem is proved. Q.E.D.

Definition: Let $\mu_n : \Sigma \rightarrow \mathcal{X}$ be finitely additive and s-bounded, $n = 1, 2, \cdots$. We say the μ_n are uniformly additive if $\lim_j \sum_{K\in[j,\infty)\cap\Delta} \mu_n(E_K) = 0$ uniformly in Δ and n whenever $\{E_K\}$ is a pairwise disjoint sequence of sets in Σ. That is, for all $\epsilon > 0$, there exists J such that for all $j \geq J$, all n, and all Δ, $\left|\sum_{K\in[j,\infty)\cap\Delta} \mu_n(E_K)\right| < \epsilon$.

Remark: When the μ_n are countably additive this is equivalent to the μ_n being uniformly countably additive.

Theorem 4.4: Let $\mu_n : \Sigma \to \mathfrak{X}$ be finitely additive and s-bounded, $n = 1, 2, \cdots$. Assume $\mu(E) = \lim_n \mu_n(E)$ exists for every E. Then the μ_n are uniformly additive.

Proof: Let $\{E_K\}$ be a sequence of disjoint sets and let $\epsilon > 0$. Let $\nu_n = \mathcal{P}(\eta) \to \mathfrak{X}$ be defined by $\nu_n(\Delta) = (\mu_n - \mu)\left(\bigcup_{i\in\Delta} E_i\right)$. By Corollary 4.3 μ is s-bounded, so the ν_n satisfy the hypothesis of Theorem 4.1. Hence there exists an integer N such that:

(1) $\left|\sum_{K\in\Delta} (\mu_n - \mu)(E_K)\right| < \epsilon$, for all $n \geq N$ and all Δ.

Since μ is s-bounded, it follows from Theorem 2.4 that $\sum_{K=1}^{\infty} \mu(E_K)$ converges unconditionally, so by Theorem 1.2 there exists I_1 such that:

(2) $\left|\sum_{K\in\Delta\cap[I_1,\infty)} \mu(E_K)\right| < \epsilon$ for all Δ .

Combining inequalities (1) and (2) we have that for all $n \geq N$ and for all Δ , $\left|\sum_{K\in\Delta\cap[I_1,\infty)} \mu_n(E_K)\right| \leq \left|\sum_{K\in\Delta\cap[I_1,\infty)} \mu(E_K)\right| + \left|\sum_{K\in\Delta\cap[I_1,\infty]} (\mu_n-\mu)(E_K)\right| < 2\epsilon$.

Each μ_n is s-bounded, so there also exists I_2 such that $\left|\sum_{K\in\Delta\cap[I_2,\infty)} \mu_n(E_K)\right| < \epsilon$ for all Δ and all $n < N$. Consequently $\left|\sum_{K\in\Delta\cap[I_1+I_2,\infty)} \mu_n(E_K)\right| < 2\epsilon$ for all n and Δ . Since ϵ was chosen arbitrarily, this proves that the μ_n are uniformly additive.

Q.E.D.

Remark: Corollary 4.3 and Theorem 4.4 are the intended generalizations of the Nikodym theorem.

Corollary 4.5: Let $\mu_n : \Sigma \to R$ be bounded and finitely additive, $n = 1, 2, \cdots$. If $\mu(E) = \lim_n \mu_n(E)$ exists for all E , then the μ_n are uniformly additive.

Proof: It follows from Theorem 2.2 that the μ_n are s-bounded, so the conclusion follows immediately from Theorem 4.4. Q.E.D.

Corollary 4.6 (Nikodym): Let $\mu_n : \Sigma \to \mathcal{X}$ be countably additive, $n = 1, 2, \cdots$. If $\mu(E) = \lim_n \mu_n(E)$ exists for every E , then μ is countably additive and the countable additivity of the μ_n is uniform.

Proof: Let $\epsilon > 0$ and let $\{E_K\}$ be a sequence of disjoint sets. By Corollary 4.3, μ is s-bounded, so $\sum_{K=1}^{\infty} \mu(E_K)$ exists, hence there is an I_1 such that for all $i \geq I_1$:

(1) $\left| \sum_{K=1}^{\infty} \mu(E_K) - \sum_{K=1}^{i} \mu(E_K) \right| < \frac{\epsilon}{2}$.

By Theorem 4.4, the μ_n are uniformly countably additive, so there exists I_2 such that for all n and all $i \geq I_2$, $\left| \sum_{K=i}^{\infty} \mu_n(E_K) \right| < \frac{\epsilon}{2}$ which means $\left| \mu_n\left(\bigcup_{K=i}^{\infty} E_K \right) \right| < \frac{\epsilon}{2}$ because each μ_n is countably additive. Therefore:

(2) $\left| \mu\left(\bigcup_{K=i}^{\infty} E_K \right) \right| = \left| \lim_n \mu_n\left(\bigcup_{K=i}^{\infty} E_K \right) \right| \leq \frac{\epsilon}{2}$ for all $i \geq I_2$.

Let $I = \max\{I_1, I_2\}$. Obviously μ is finitely additive, so:

(3) $\mu\left(\bigcup_{K=1}^{I} E_K \right) = \sum_{K=1}^{I} \mu(E_K)$.

Combining inequalities (1), (2), and (3) we have $\left| \mu\left(\bigcup_{K=1}^{\infty} E_K \right) - \sum_{K=1}^{\infty} \mu(E_K) \right| \leq \left| \mu\left(\bigcup_{K=I+1}^{\infty} E_K \right) \right| + \left| \mu\left(\bigcup_{K=1}^{I} E_K \right) - \sum_{K=1}^{I} \mu(E_K) \right| + \left| \sum_{K=1}^{I} \mu(E_K) - \sum_{K=1}^{\infty} \mu(E_K) \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ was arbitrary, this implies that $\mu\left(\bigcup_{K=1}^{\infty} E_K \right) = \sum_{K=1}^{\infty} \mu(E_K)$. Q.E.D.

Definition: Let $\mu : \Sigma \to \mathfrak{X}$ and $\nu : \Sigma \to R^+$. We say μ is absolutely continuous with respect to ν if for every $\epsilon > 0$, there exists $\delta > 0$ so that $\nu(E) < \delta$ implies $|\mu(E)| < \epsilon$ for all E in Σ. If $\mu_n : \Sigma \to \mathfrak{X}$, we say the μ_n are uniformly absolutely continuous with respect to ν if for all $\epsilon > 0$ there is a $\delta > 0$ such that $\nu(E) < \delta$ implies $|\mu_n(E)| < \epsilon$ for all E and n.

Remark: The next theorem is a generalization of the Vitali-Hahn-Saks theorem, and is the most important result of this dissertation.

Theorem 4.7: Let $\mu_n : \Sigma \to \mathfrak{X}$ be finitely additive and s-bounded, $n = 1, 2, \cdots$. Assume that $\lim_n \mu_n(E)$ exists for every E. Let ν be a non-negative (possibly infinite) finitely additive set function defined on Σ. If μ_n is absolutely continuous with respect to ν for each n, then the μ_n are uniformly absolutely continuous with respect to ν.

Remark: If ν were assumed to be bounded, then the μ_n would automatically be s-bounded.

Proof: Deny the conclusion. Then there exists an $\epsilon > 0$

such that for every $\delta > 0$, there exists an E, such that $\nu(E) < \delta$ and $|\mu_n(E)| > \epsilon$ for some n. Let E_1, μ_{n_1} be such that $\nu(E_1) < 1$ and $|\mu_{n_1}(E_1)| > \epsilon$. Let δ_2 be chosen so that $\nu(E) < \delta_2$ implies that $|\mu_{n_1}(E)| < \frac{\epsilon}{2^4}$. Obtain E_2, n_2 such that $n_2 > n_1$, $\nu(E_2) < \delta_2$, and $|\mu_{n_2}(E_2)| > \epsilon$. Then for all $F \subset E_2$, $|\mu_{n_1}(F)| < \frac{\epsilon}{2^4}$.

Now assume that $E_1 \cdots E_K$, $n_1 \cdots n_K$ have been chosen. Let δ_{K+1} be such that for all $i = 1 \cdots K$, $\nu(E) < \delta_{K+1}$ implies $|\mu_{n_i}(E)| < \frac{\epsilon}{2^{K+3}}$. Now choose E_{K+1} and n_{K+1} so that $\nu(E_{K+1}) < \delta_{K+1}$ $n_{K+1} > n_K$ and $|\mu_{n_{K+1}}(E_{K+1})| > \epsilon$. Then for all $F \subset E_{K+1}$ and for all $i = 1 \cdots K$, $|\mu_{n_i}(F)| < \frac{\epsilon}{2^{K+3}}$.

Resubscripting for simplification, let $\mu_i = \mu_{n_i}$. We then have that:

(1) for all i, $|\mu_i(E_i)| > \epsilon$ and for all $j = 1 \cdots i-1$ and all $F \subset E_i$, $|\mu_j(F)| < \frac{\epsilon}{2^{i+2}}$.

Let $F_1 = E_2$, $i_1 = 1$, and assume there exists an $i_2 > 2$ such that $|\mu_{i_2}(F_1 \cap E_{i_2})| \geq \frac{\epsilon}{4}$. Let $F_2 = F_1 - E_{i_2}$. Assume $F_1 \cdots F_K$ have been chosen in this manner, along with $i_2 \cdots i_K$.

Assume there exists $i_{K+1} > i_K$ such that $\left|\mu_{i_{K+1}}(F_K \cap E_{i_{K+1}})\right| \geq \frac{\epsilon}{4}$.

Then define $F_{K+1} = F_K - E_{i_{K+1}}$. If this process continues we obtain a countable collection of sets F_K such that

$\left|\mu_{i_{K+1}}(F_K \cap F_{K+1})\right| = \left|\mu_{i_{K+1}}(F_K \cap E_{i_{K+1}})\right| \geq \frac{\epsilon}{4}$, and

$\left|\mu_{i_K}(F_K - F_{K+1})\right| = \left|\mu_{i_K}(F_K \cap E_{i_{K+1}})\right| < \frac{\epsilon}{2^{i_{K+1}+2}} < \frac{\epsilon}{8}$

because of (1) and the fact that $F_K \cap E_{i_{K+1}} \subset E_{i_{K+1}}$.

Therefore $\left|(\mu_{i_{K+1}} - \mu_{i_K})(F_K - F_{K+1})\right| \geq \frac{\epsilon}{4} - \frac{\epsilon}{4} = \frac{\epsilon}{8}$.

Note that $\{F_K - F_{K+1}\}$ is a sequence of disjoint sets, and $\left\{\sum_{K \epsilon \Delta}(\mu_{i_{j+1}} - \mu_{i_j})(F_K - F_{K+1})\right\}_{j=1}^{\infty}$ does not converge to zero uniformly in Δ , which contradicts Corollary 4.2.

$\therefore$

Therefore the process of choosing the F_K and i_K has to stop, so there exists F_K , $i_K \geq 2$ such that for all $j > i_K$, $\left|\mu_j(F_K \cap E_j)\right| < \frac{\epsilon}{4}$. Let $P_1 = i_K$, $H_1 = F_K$, $\mu_i^{(1)} = \mu_{P_1+i}$, and $E_i^{(1)} = E_{P_1+i} - H_1$. Then the following three results obtain.

I. $\left|\mu_1(H_1)\right| < \frac{\epsilon}{16}$.

Since $H_1 \subset E_2$, it follows from (1) that $\left|\mu_1(H_1)\right| < \frac{\epsilon}{2^{2+2}} = \frac{\epsilon}{16}$.

II. $\left|\mu_2(H_1)\right| > \epsilon - \frac{\epsilon}{4}$.

We have that $E_2 = F_1 = (F_1 - F_2) \cup (F_2 - F_3) \cup \cdots \cup (F_{K-1} - F_K) \cup F_K$.

$|\mu_2(E_2)| > \epsilon$ and $|\mu_2(F_1 - F_2)| = |\mu_2(F_1 \cap E_{i_2})| < \frac{\epsilon}{2^{i_2+2}} < \frac{\epsilon}{8}$,

since $F_1 \cap E_{i_2} \subset E_{i_2}$ and $2 < i_2$. $|\mu_2(F_1 - F_2)| = |\mu_2(F_1 \cap E_{i_2})| <$

$\frac{\epsilon}{2^{i_3+2}} < \frac{\epsilon}{16}$ because $i_3 > i_2 > 2$, hence $i_3 > 4$. Thus for

$j = 1 \cdots K-1$, $|\mu_2(F_j - F_{j+1})| = |\mu_2(F_j \cap E_{i_{j+1}})| < \frac{\epsilon}{2^{j+2}}$, and

$\epsilon < |\mu_2(E_2)| \leq |\mu_2(F_1 - F_2)| + \cdots + |\mu_2(F_{K-1} - F_K)| + |\mu_2(F_K)| <$

$\frac{\epsilon}{8} + \frac{\epsilon}{16} + \frac{\epsilon}{32} + \cdots + |\mu_2(H_1)|$, since $H_1 = F_K$. Therefore

$|\mu_2(H_1)| > \epsilon - (\frac{\epsilon}{8} + \frac{\epsilon}{16} + \frac{\epsilon}{32} + \cdots) = \epsilon - \frac{\epsilon}{4}$.

III. $|\mu_i^{(1)}(E_i^{(1)})| > \epsilon - \frac{\epsilon}{4}$.

Note that $E_{p_1+i} = (E_{p_1+i} - H_1) \cup (E_{p_1+i} \cap H_1)$. H_1 was chosen so that for all $j > p_1$, $|\mu_j(E_j \cap H_1)| < \frac{\epsilon}{4}$. Hence

$\epsilon < |\mu_{p_1+i}(E_{p_1+i})| \leq |\mu_{p_1+i}(E_{p_1+i} - H_1)| + |\mu_{p_1+i}(E_{p_1+i} \cap H_1)| <$

$|\mu_i^{(1)}(E_i^{(1)})| + \frac{\epsilon}{4}$. Thus $\epsilon - \frac{\epsilon}{4} < |\mu_i^{(1)}(E_i^{(1)})|$.

<u>Note</u>: From I and II we have that $|(\mu_2 - \mu_1)(H_1)| > \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{16}$.

Let $F_1^{(1)} = E_1^{(1)}$. Assume that there exists $i_2 > 1$ such that $|\mu_{i_2}^{(1)}(E_{i_2}^{(1)})| \geq \frac{\epsilon}{8}$, and let $F_2^{(1)} = F_1^{(1)} - E_{i_2}^{(1)}$. If we could continue to choose $\{F_K^{(1)}\}$ indefinitely, then we would obtain a contradiction the same as before. So there exists $F_K^{(1)}$

and $i_K \geq 2$ so that for all $j > i_K$, $\left|\mu_j^{(1)}(F_K^{(1)} \cap E_j^{(1)})\right| < \frac{\epsilon}{8}$.

Let $H_2 = F_K^{(1)}$, $P_2 = i_K$, $\mu_i^{(2)} = \mu_{P_2+i}^{(1)} = \mu_{P_1+P_2+i}$,

and $E_i^{(2)} = E_{P_2+i}^{(1)} - H_2 = E_{P_1+P_2+i} - (H_1 \cup H_2)$.

Then $H_1 \cap H_2 = \square$ because $H_2 \subset E_1^{(1)}$ and $E_1^{(1)} \cap H_1 = \square$. (Recall the construction of $E_1^{(1)}$.) The following results are then obtained.

I'. $|\mu_2(H_2)| < \frac{\epsilon}{32}$.

Since $H_2 \subset E_1^{(1)} \subset E_{P_1+1}$ and $P_1 + 1 > 2$, we have that

$$|\mu_2(H_2)| < \frac{\epsilon}{2^{P_1+1+2}} \leq \frac{\epsilon}{32} .$$

II'. $\left|\mu_1^{(1)}(H_2)\right| > \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{8}$.

$(F_1^{(1)} - F_2^{(1)}) \cup (F_2^{(1)} - F_3^{(1)}) \cup \cdots \cup (F_{K-1}^{(1)} - F_K^{(1)}) \cup F_K = F_1^{(1)} = E_1^{(1)}$, so $\epsilon - \frac{\epsilon}{4} < \left|\mu_1^{(1)}(E_1^{(1)})\right| \leq \left|\mu_1^{(1)}(F_1^{(1)} - F_2^{(1)})\right| + \cdots + \left|\mu_1^{(1)}(F_{K-1}^{(1)} - F_K^{(1)})\right| + \left|\mu_1^{(1)}(H_2)\right|$, since $H_2 = F_K^{(1)}$. Since $F_1^{(1)} \cap E_{i_2}^{(1)} \subset E_{i_2}^{(1)} \subset E_{P_1+i_2}$ and $i_2 > 1$, we have that $\left|\mu_1^{(1)}(F_1^{(1)} - F_2^{(1)})\right| = \left|\mu_1^{(1)}(F_1^{(1)} \cap E_{i_2}^{(1)})\right| < \frac{\epsilon}{2^{P_1+i_2+2}} < \frac{\epsilon}{2^{i+3}}$. Likewise $\left|\mu_1^{(1)}(F_j^{(1)} - F_{j+1}^{(1)})\right| \leq \left|\mu_1^{(1)}(F_j^{(1)} \cap E_{i_{j+1}}^{(1)})\right| < \frac{\epsilon}{2^{P_1+i_{j+1}+2}} < \frac{\epsilon}{2^{j+3}}$ because $i_{j+1} \geq j$ for all $j = 1, 2, \cdots K-1$. Hence $\epsilon - \frac{\epsilon}{4} < \frac{\epsilon}{16} + \frac{\epsilon}{32} + \cdots +$

$\left|\mu_1^{(1)}(H_2)\right| \le \frac{\epsilon}{8} + \left|\mu_1^{(1)}(H_2)\right|$, and it follows that $\left|\mu_1^{(1)}(H_2)\right| >$ $\epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{8}$.

III'. $\left|\mu_i^{(2)}(E_i^{(2)})\right| > \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{8}$, for all i .

Since $H_1 \cap H_2 = \square$, the sets $(E_{p_1+p_2+i} \cap H_1)$, $(E_{p_1+p_2+i} \cap H_2)$, and $(E_{p_1+p_2+i} - (H_1 \cup H_2))$ are disjoint. Therefore $\epsilon < \left|\mu_i^{(2)}(E_{p_1+p_2+i})\right| \le \left|\mu_i^{(2)}(E_{p_1+p_2+i} \cap H_1)\right| + \left|\mu_i^{(2)}(E_{p_1+p_2+i} \cap H_2)\right| + \left|\mu_i^{(2)}(E_{p_1+p_2+i} - (H_1 \cup H_2))\right|$. Note that $\left|\mu_i^{(2)}(E_{p_1+p_2+i} \cap H_1)\right| < \frac{\epsilon}{4}$ and $\left|\mu_i^{(2)}(E_{p_1+p_2+i} \cap H_2)\right| < \frac{\epsilon}{8}$ because of the construction of H_1 and H_2 . Also, by definition $\left|\mu_i^{(2)}(E_{p_1+p_2+i} - (H_1 \cup H_2))\right| = \left|\mu_i^{(2)}(E_i^{(2)})\right|$. Therefore $\left|\mu_i^{(2)}(E_i^{(2)})\right| = \left|\mu_i^{(2)}(E_{p_1+p_2+i} - (H_1 \cup H_2))\right| > \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{8}$.

<u>Note:</u> From I' and II' we obtain $\left|(\mu_1^{(1)} - \mu_2)(H_2)\right| > \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{8} - \frac{\epsilon}{32}$.

We now proceed to the general induction step.

Let $K \ge 2$, $\mu_i^{(0)} = \mu_{i+1}$, $E_i^{(0)} = E_i$. Assume that the following statements are true. For each $j = 1 \cdots K$, H_j and $p_j \ge 2$ are defined and $E_i^{(j)} = E_{p_j+i}^{(j-1)} - H_j$, . $\mu_i^{(1)} = \mu_{p_1+i}$, $\left|\mu_i^{(j)}(E_i^{(j)})\right| > \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{8} - \cdots - \frac{\epsilon}{2^{j+1}}$, $H_p \cap H_j = \square$ if $p = 1 \cdots K$,

$p \neq j$ and if $j \geq 2$, $\mu_i^{(j)} = \mu_{p_j+i}^{(j-1)}$, $|\mu_1^{(j-2)}(H_j)| < \frac{\epsilon}{32}$,

$|\mu_1^{(j-1)}(H_j)| > \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{8} - \cdots - \frac{\epsilon}{2^{j+1}}$.

All of the above statements are true when $K=2$. I will now show that their being true for K implies they are true for $K+1$.

Let $F_1^{(K)} = E_1^{(K)}$, $i_1 = 2$. Assume there exists $i_2 > i_1$ such that $|\mu_{i_2}^{(K)}(F_1^{(K)} \cap E_{i_2}^{(K)})| \geq \frac{\epsilon}{2^{K+2}}$, and let $F_2^{(K)} = F_1^{(K)} - E_{i_2}^{(K)}$. If there exists $i_3 > i_2$ so that $|\mu_{i_3}^{(K)}(F_2^{(K)} \cap E_{i_3}^{(K)})| \geq \frac{\epsilon}{2^{K+2}}$, let $F_3^{(K)} = F_2^{(K)} - E_{i_3}^{(K)}$. If this process of choosing i_j and F_j should go on indefinitely we obtain a contradiction because for all j , $|\mu_{i_j}^{(K)}(F_{j-1}^{(K)} - F_j^{(K)})| = |\mu_{i_j}^{(K)}(F_{j-1}^{(K)} \cap E_{i_j}^{(K)})| \geq \frac{\epsilon}{2^{K+2}}$, whereas $F_{j-1}^{(K)} \cap E_{i_j}^{(K)} \subset E_{i_j}^{(K)} \subset E_{p_1+\cdots+p_K+i_j}$ and $i_j > 1$ imply that $|\mu_{i_{j-1}}^{(K)}(F_{j-1}^{(K)} \cap E_{i_j}^{(K)})| < \frac{\epsilon}{2^{p_1+\cdots+p_K+i_j+2}} < \frac{\epsilon}{2^{K+3}}$.

Combining these two inequalities we see that for all j ,

$|(\mu_{i_j}^{(K)} - \mu_{i_{j-1}}^{(K)})(F_{j-1}^{(K)} - F_j^{(K)})| > \frac{\epsilon}{2^{K+3}}$. Therefore

$\left\{ \sum_{j \in \Delta} (\mu_{i_n}^{(K)} - \mu_{i_{n-1}}^{(K)})(F_{j-1}^{(K)} - F_j^{(K)}) \right\}_{j=1}^{\infty}$ does not converge to zero uniformly in Δ . This contradicts Corollary 4.2.

Hence there exist $F_t^{(K)}$ and $i_t \geq 2$ such that for all $j > i_t$, $\left|\mu_j^{(K)}(F_t^{(K)} \cap E_j^{(K)})\right| < \frac{\epsilon}{2^{K+3}}$. Let $P_{K+1} = i_t$, $H_{K+1} = F_t^{(K)}$, $\mu_i^{(K+1)} = \mu^{(K)}_{P_{K+1}+i} = \mu_{P_1 + \cdots + P_K + P_{K+1} + i}$, $E_i^{(K+1)} = E^{(K)}_{P_{K+1}+i} - H_{K+1} = E_{P_1 + \cdots + P_K + P_{K+1} + i} - \left(\bigcup_{j=1}^{K+1} H_j\right)$.

In order to show the induction step is valid, I must prove the following four statements.

I''. For all $j = 1 \cdots K$, $H_j \cap H_{K+1} = \square$.

$H_{K+1} \subset E_1^{(K)} = F_1^{(K)}$ by construction, and $E_1^{(K)} = E_{P_1 + \cdots + P_K+1} - \left(\bigcup_{j=1}^{K} H_j\right)$. Hence $H_{K+1} \cap \left(\bigcup_{j=1}^{K} H_j\right) = \square$.

II''. $\left|\mu_1^{(K-1)}(H_{K+1})\right| < \frac{\epsilon}{32}$.

Since $H_{K+1} \subset E_{P_1 + \cdots + P_K + 1}$, $\left|\mu_1^{(K-1)}(H_{K+1})\right| = \left|\mu_{P_1 + \cdots + P_{K-1}+1}(H_{K+1})\right| < \frac{\epsilon}{2^{P_1 + \cdots + P_K + 2}} \leq \frac{\epsilon}{32}$ because $P_1 \geq 2$.

III''. $\left|\mu_1^{(K)}(H_{K+1})\right| > \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{8} - \cdots - \frac{\epsilon}{2^{K+2}}$.

Note that $E_1^{(K)} = (F_1^{(K)} - F_2^{(K)}) \cup (F_2^{(K)} - F_3^{(K)}) \cup \cdots \cup (F_{t-1}^{(K)} - F_t^{(K)}) \cup F_t^{(K)}$.

Also $\left|\mu_1^{(K)}(F_1^{(K)} - F_2^{(K)})\right| = \left|\mu_1^{(K)}(F_1^{(K)} \cap E_{i_2}^{(K)})\right| < \frac{\epsilon}{2^{P_1 + \cdots + P_K + i_2 + 2}} < \frac{\epsilon}{2^{K+2+1}}$ because $F_2^{(K)} \cap E_{i_2}^{(K)} \subset E_{P_1 + \cdots + P_K + i_2}$ and $1 < i_2$.

Likewise $\left|\mu_1^{(K)}\left(F_2^{(K)}-F_3^{(K)}\right)\right|=\left|\mu_1^{(K)}\left(F_2^{(K)}\cap E_{i_3}^{(K)}\right)\right|<\frac{\epsilon}{2^{P_1+\cdots+P_K+i_3+2}}<$ $\frac{\epsilon}{2^{K+2+2}}$ because $F_2^{(K)}\cap E_{i_3}^{(K)}\subset E_{P_1+\cdots+P_K+i_3}$ and $2<i_3$.

In the same manner, for $j=1\cdots t-1$ we have $\left|\mu_1^{(K)}\left(F_j^{(K)}-F_{j+1}^{(K)}\right)\right|=$ $\left|\mu_1^{(K)}\left(F_j^{(K)}\cap E_{i_{j+1}}^{(K)}\right)\right|<\frac{\epsilon}{2^{P_1+\cdots+P_K+i_{j+1}+2}}<\frac{\epsilon}{2^{K+2+j}}$ because $j<i_{j+1}$. Therefore $\epsilon-\frac{\epsilon}{4}-\frac{\epsilon}{8}-\cdots-\frac{\epsilon}{2^{K+1}}<\left|\mu_1^{(K)}\left(E_1^{(K)}\right)\right|\leq$ $\frac{\epsilon}{2^{K+2+1}}+\frac{\epsilon}{2^{K+2+2}}+\cdots+\frac{\epsilon}{2^{K+2+(t-1)}}+\left|\mu_1^{(K)}(H_{K+1})\right|$; the first inequality follows from the induction hypothesis and the second inequality results from the definition of H_{K+1} and the inequalities in the preceding sentences. Therefore,

$$\epsilon-\frac{\epsilon}{4}-\frac{\epsilon}{8}-\cdots-\frac{\epsilon}{2^{K+1}}-\left(\frac{\epsilon}{2^{K+2+1}}+\cdots+\frac{\epsilon}{2^{K+2+(t-1)}}\right)<\left|\mu_1^{(K)}(H_{K+1})\right|.$$

Since $\frac{\epsilon}{2^{K+2+1}}+\cdots+\frac{\epsilon}{2^{K+2+(t-1)}}<\frac{\epsilon}{2^{K+2}}$, we have

$$\epsilon-\frac{\epsilon}{4}-\frac{\epsilon}{8}-\cdots-\frac{\epsilon}{2^{K+1}}-\frac{\epsilon}{2^{K+2}}<\left|\mu_1^{(K)}(H_{K+1})\right|.$$

IV''. For all i, $\left|\mu_i^{(K+1)}\left(E_i^{(K+1)}\right)\right|>\epsilon-\frac{\epsilon}{4}-\cdots-\frac{\epsilon}{2^{K+2}}$.

$$\left|\mu_{P_{K+1}+i}^{(K)}\left(E_{P_{K+1}+i}^{(K)}-H_{K+1}\right)\right|+\left|\mu_{P_{K+1}+i}^{(K)}\left(E_{P_{K+1}+i}^{(K)}\cap H_{K+1}\right)\right|\geq\left|\mu_{P_{K+1}+i}^{(K)}\left(E_{P_{K+1}+i}^{(K)}\right)\right|>$$

$\epsilon-\frac{\epsilon}{4}-\cdots-\frac{\epsilon}{2^{K+1}}$, because of the induction hypothesis.

As a result of the construction of H_{K+1} and because $P_{K+1}+i>P_{K+1}$, we have $\left|\mu_{P_{K+1}+i}^{(K)}\left(E_{P_{K+1}+i}^{(K)}\cap H_{K+1}\right)\right|<\frac{\epsilon}{2^{K+2}}$. Thus

$$\left|\mu_i^{(K+1)}\left(E_i^{(K+1)}\right)\right|=\left|\mu_{P_{K+1}+i}^{(K)}\left(E_{P_{K+1}+i}^{(K)}-H_{K+1}\right)\right|>\epsilon-\frac{\epsilon}{4}-\cdots-\frac{\epsilon}{2^{K+1}}-\frac{\epsilon}{2^{K+2}}.$$

Note: It follows from II'' and III'' that

$$\left|\left(\mu_1^{(K)} - \mu_1^{(K-1)}\right)(H_{K+1})\right| > \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{8} - \cdots - \frac{\epsilon}{2^{K+2}} - \frac{\epsilon}{32} \quad .$$

Therefore, by induction we may define infinite collections $\{H_K\}$ and $\{\mu_1^{(K)}\}$ such that for each $K \geq 2$, $\left|\left(\mu_1^{(K-1)} - \mu_1^{(K-2)}\right)(H_K)\right| > \epsilon - \frac{\epsilon}{4} - \frac{\epsilon}{8} - \cdots - \frac{\epsilon}{2^{K+2}} - \frac{\epsilon}{32} > \epsilon - \frac{\epsilon}{2} - \frac{\epsilon}{32} > \frac{\epsilon}{16}$. $\{H_K\}$ is also a sequence of disjoint sets. Letting $\nu_2 = \mu_2$, $\nu_3 = \mu_1^{(1)}$, $\nu_n = \mu_1^{(n-2)}$, then $\left|(\nu_{n+1} - \nu_n)(H_n)\right| > \frac{\epsilon}{16}$ for all $n \geq 2$. So $\left\{\sum_{K \in \Delta}(\nu_{n+1} - \nu_n)(H_K)\right\}_{n=2}^{\infty}$ does not converge to zero uniformly in Δ , which contradicts Corollary 4.2. This concludes the proof of the theorem. Q.E.D.

Definition: A linear topological space is said to be locally convex if it has a base consisting of convex sets. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be such a base. Let $\mathcal{F} = \{U_\alpha \in \mathcal{U} : 0 \in U_\alpha\}$. For each U_α in $\mathcal{F}$, define $I_\alpha(x) = \{a : a > 0, \frac{x}{a} \in U_\alpha\}$ and $\rho_\alpha(x) = \inf\{a : a \in I_\alpha(x)\}$. Then ρ_α is a pseudo-norm for the space $\mathfrak{X}$, and has the following properties.

(a) $\rho_\alpha(x) \geq 0$

(b) $\rho_\alpha(x) < +\infty$

(c) $\rho_\alpha(ax) = a\rho_\alpha(x)$

(d) if $x \in U_\alpha$, then $\rho_\alpha(x) \leq 1$

(e) $\rho_\alpha(x+y) \leq \rho_\alpha(x) + \rho_\alpha(y)$

That is, ρ_α has all the properties of a norm except that it may be possible to find an x such that $x \neq 0$ and $\rho_\alpha(x) = 0$.

In the theorems of this chapter the fact that $|x| = 0$ implies $x = 0$ was never used, and with appropriate modifications the theorems are true if the set functions take values in a locally convex linear topological space. First of all, one has to define s-bounded and absolute continuity differently.

Definition: Let $\mathfrak{X}$ be a locally convex linear topological space, and let $\{\rho_\alpha : \alpha \in A\}$ be a family of pseudo-norms that determine the topology of $\mathfrak{X}$. Let $\mu: \Sigma \to \mathfrak{X}$ and $\nu: \Sigma \to R^+$ be set functions. We say μ is strongly bounded (s-bounded) if $\lim_K \mu(E_K) = 0$ whenever $\{E_K\}$ is a sequence of disjoint sets. μ is absolutely continuous with respect to ν if for every α and ϵ, there exists $\delta > 0$ such that $\nu(E) < \delta$ implies $\rho_\alpha(\mu(E)) < \epsilon$.

The topology on $\mathfrak{X}$ determined by a single pseudo-norm α is not necessarily complete. We then let $\hat{\mathfrak{X}}_\alpha$ be the completion of $\mathfrak{X}$, and if $\mu : \Sigma \to \mathfrak{X}$ is s-bounded, we consider μ to have values in $\hat{\mathfrak{X}}_\alpha$, and it is then true that $\sum_{n=1}^{\infty} \mu(E_n)$ converges unconditionally in $\hat{\mathfrak{X}}_\alpha$ if $\{E_n\}$ is a sequence of disjoint sets. The statements of Theorem 4.1 and Corollary 4.2 do not change if $\mathfrak{X}$ is locally convex, except that $\sum_{n \in \Delta} \mu_n(t)$ is considered to be an element of $\hat{\mathfrak{X}}_\alpha$ for a fixed α. Taking into consideration the new definition of absolute continuity, the extensions of the Vitali-Hahn-Saks and Nikodym theorems for finitely additive vector measures are valid when the range of the measures is a locally convex linear topological space.

CHAPTER 5

A COUNTEREXAMPLE

In this chapter an example is given to show that weak convergence does not imply strong convergence in the space of countably additive set functions.

Let $\mu_n : \mathcal{P}(n) \to R$ be countably additive, and assume that for all Δ, $\lim_n \mu_n(\Delta) = 0$. Then from Theorem 4.1, we have that $\lim_n \mu_n(\Delta) = 0$ uniformly in Δ, which in turn implies that $\lim_n |\mu_n| = 0$, where $|\mu_n|$ is the total variation of μ_n.

Since Theorem 4.1 was very important in proving theorems about set functions from a σ-algebra to a Banach space, it is reasonable to pose the following question: if $\mu_n : \Sigma \to R$ and if $\lim_n \mu_n(E) = 0$ for every $E \in \Sigma$, does it follow that $\lim_n |\mu_n| = 0$? The answer to this question is, "No," as the following counterexample shows.

Counterexample: This counterexample will show that if $\mu_n : \Sigma \to R$ are countably additive and if $\lim_n \mu_n(E) = 0$ for all E, then it does not follow that $\lim_n |\mu_n| = 0$.

Construction: Let Σ be the Lebesgue measurable subsets of $(0, 1)$, and let λ be Lebesgue measure restricted to Σ.

Let f_n be defined:

$$f_n(x) = \begin{cases} 1 & \text{if } x \text{ is in } \left(\frac{a}{2^n}, \frac{a+1}{2^n}\right), \ 0 \le a < 2^n, \ a \text{ even} \\ -1 & \text{if } x \text{ is in } \left(\frac{a}{2^n}, \frac{a+1}{2^n}\right), \ 0 \le a < 2^n, \ a \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Let $\mu_n(E) = \int_E f_n \, d\lambda$. Then for all $\delta > 0$, $\lambda(E) < \delta$ implies $\left|\int_E f_n \, d\lambda\right| < \delta$, which is equivalent to $|\mu_n(E)| < \delta$. Note that if $K > n$, then $\mu_K\left[\left(\frac{a}{2^n}, \frac{a+1}{2^n}\right)\right] = 0$.

First it will be shown that if $\mathcal{O}$ is an open interval, then $\lim_n \mu_n(\mathcal{O}) = 0$. Let $\epsilon > 0$, and choose n so that $\frac{1}{2^n} < \epsilon$. Let $b = \min\left\{a \in \mathcal{N} : \frac{a}{2^n} \in \mathcal{O}\right\}$, $c = \max\left\{a \in \mathcal{N} : \frac{a}{2^n} \in \mathcal{O}\right\}$.

Then $\lambda\left(\mathcal{O} - \cup\left\{\left(\frac{a}{2^n}, \frac{a+1}{2^n}\right) : b \le a < c\right\}\right) < 2\epsilon$, and this inequality combined with the statements in the last two sentences of the preceding paragraph give us that $|\mu_K(\mathcal{O})| < 2\epsilon$ for all $K > n$.

Since ϵ was chosen arbitrarily, the conclusion follows.

Now let V be an open set in Σ and let $\epsilon > 0$. There exists a countable collection of disjoint open intervals $\{\mathcal{O}_i\}$ such that $V = \bigcup_{i=1}^{\infty} \mathcal{O}_i$. (I am considering the null set to be an open interval.) Then $\lambda(V) = \sum_{i=1}^{\infty} \lambda(\mathcal{O}_i)$, and, since $\lambda(V) < \infty$, there exists N such that $\lambda\left(V - \bigcup_{i=1}^{N} \mathcal{O}_i\right) < \epsilon$; hence for all n, $\left|\mu_n(V) - \mu_n\left(\bigcup_{i=1}^{N} \mathcal{O}_i\right)\right| < \epsilon$. For all $i = 1, \cdots, N$, let T_i be such that for all $K > T_i$, $\left|\mu_K(\mathcal{O}_i)\right| < \frac{\epsilon}{N}$. Then letting $T = \max\{T_i : i = 1, \cdots, N\}$, we have that for every $K > T$, $\left|\mu_K(V)\right| \leq \left|\mu_K(V) - \mu_K\left(\bigcup_{i=1}^{N} \mathcal{O}_i\right)\right| + \left|\sum_{i=1}^{N} \mu_K(\mathcal{O}_i)\right| < \epsilon + N\left(\frac{\epsilon}{N}\right) = 2\epsilon$. Therefore we have that $\lim_n \mu_n(V) = 0$ for all open sets V.

Let E be a set in Σ, and let $\epsilon > 0$. There exists an open set V so that $E \subset V$ and $\lambda(V - E) < \epsilon$, and it follows that $\mu_n(V - E) < \epsilon$ for all n. By the result of the last paragraph there exists T such that for all $n > T$, $\left|\mu_n(V)\right| < \epsilon$, hence $\left|\mu_n(E)\right| \leq \left|\mu_n(V)\right| + \left|\mu_n(V - E)\right| < 2\epsilon$. This implies that $\lim_n \mu_n(E) = 0$ for every E in Σ.

The μ_n are countably additive. so they satisfy the hypothesis of the counterexample. However, $|\mu_n|(0,1) = \int_0^1 |f_n| \, d\lambda = 1$ for all n, so $\{|\mu_n|\}$ does not converge to zero.

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BIOGRAPHICAL SKETCH

Robert Jewett was born in Portsmouth, Ohio, on August 20, 1945. When he was eleven years old his family moved to Fort Myers, Florida, where he graduated from high school in 1963. In the Fall of the same year he went to the University of Florida on a golf scholarship, and played on the golf team for two years. In 1967 he received his Bachelor of Science degree in Math, and from that time on has been working toward his doctor's degree. He is a member of the American Mathematical Society.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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This dissertation was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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