A NEW APPROACH TO MOTOR CALCULUS AND RIGID BODY DYNAMICS WITH APPLICATION TO SERIAL OPEN-LOOP CHAINS

By

GILBERT H. LOVELL III

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1986

ACKNOWLEDGMENTS

I would like to express my gratitude to all those who have helped make this work possible.

I am especially indebted to my advisor, Dr. Joseph Duffy, who has been a constant source of inspiration.

The many discussions with the faculty and students of the Center for Intelligent Machines and Robotics have been most helpful. Special thanks are due to Dr. Ralph Selfridge, Dr. Gary Matthew, Dr. Renatus Ziegler, Sabri Tosunoglu, Resit Soylu, Harvey Lipkin, and Mark Thomas.

Finally, I am grateful for the excellent typing by Ms. Carole Boone.

TABLE OF CONTENTS

Abstract of Dissertation Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

A NEW APPROACH TO MOTOR CALCULUS
AND RIGID BODY DYNAMICS
WITH APPLICATION TO SERIAL OPEN-LOOP CHAINS

By

GILBERT H. LOVELL III

May 1986

Chairman: Dr. Joseph Duffy
Major Department: Mechanical Engineering

Motor calculus is a mathematical system that is quite analogous to vector calculus and is particularly well-suited for rigid body dynamics. This work presents a new approach to motor calculus and its application to dynamics. As opposed to the common bivector definition, a motor is here defined to be a special kind of vector field. It is demonstrated that this novel motor definition is both useful for investigating the mathematical structure of motor calculus and for applying motor calculus to rigid body dynamics. New results include simple derivations and representations of the inertia dyadic for a rigid body and of the equations of motion for a serially connected open-loop kinematic chain.

CHAPTER I
INTRODUCTION

Motor calculus is an elegant mathematical system that is particularly well-suited to rigid body kinematics and dynamics. Briefly, a motor is a geometric entity that can be identified with the following physical entities: the velocity distribution of a rigid body, the moment distribution of a force system, and the momentum distribution of a mass system. These vector fields are respectively called the velocity motor, the wrench and the momentum motor. Rigid body dynamics can be formulated as the study of the algebraic and differential relationships among these three kinds of vector fields; these relationships are studied by means of motor calculus.

The basic ideas of motor calculus received considerable attention in the late 19th century. Perhaps the single most important work of that time was a treatise by R. S. Ball entitled <u>The Theory of Screws</u> (1900). A screw is defined as a line together with a real number called the pitch. By adjoining another real number, called the magnitude, to the screw, we obtain an entity that can be identified with a motor. It was Clifford (1873), in fact, who defined the motor as a "magnitude associated with a screw."

Ball considered, as did many other authors at that time (perhaps most notably Plücker, 1866), the dynamics of a rigid body from the standpoint of line geometry; the screw is considered the fundamental entity. The magnitude associated with a screw was incorporated through the coordinates used to represent the screw--the screw coordinates as Ball called them. Ball derived many of the dynamical relations of an initially quiescent body in terms of screw coordinates (he, in effect, did not consider inertial forces and moments due to velocity).

In motor calculus the motor as opposed to the screw is considered to be the fundamental entity. By introducing various algebraic and differential operations for motors, von Mises (1924a and 1924b) developed motor calculus in such a way that it is quite analogous to vector calculus. He also illustrated the ease with which motor calculus can be applied to the dynamics of a rigid body in general (he included the inertial forces and moments due to velocity).

For some forty years after von Mises's works, screw theory and motor calculus received little attention. The revival of these subjects, much of which has been in the area of instantaneous kinematics of rigid bodies, was initiated by Phillips and Hunt (1964), and there is continuing interest at the present time (for a good review of recent contributions, see Hunt, 1978; Bottema and Roth, 1979; and Phillips, 1984). Significant recent developments in motor calculus and its application to rigid body dynamics are due to Dimentberg (1965), Yang (1967, 1969, 1971, 1974), Woo and Freudenstein (1971), Pennock and Yang (1982), and Featherstone (1983, 1984).

It was my original intention to consider the use of motor calculus in developing efficient algorithms for dynamic analysis of serially connected open-loop kinematic chains (it should be pointed out that this has been investigated most recently by Featherstone, 1984). It turns out, however, that the emphasis of this work is more theoretical than originally intended. In the course of the research it became apparent that there were many areas of motor calculus proper and its application to rigid body dynamics that could be treated more rigorously than had been done previously.

This dissertation is essentially a rigorous development of motor calculus and its application to rigid body dynamics. An important new concept introduced is the definition of a motor as a special kind of vector field. This is in contrast to the common bivector definition (here, we use the bivector as a particular representation for a motor). The motivation for the vector field definition is that it is more tractable as regards analytical considerations of motor calculus and as regards the application of motor calculus to dynamics. Aside from investigating motor calculus proper and describing elementary dynamics via motor calculus, we also consider the dynamics of serially connected open-loop kinematic chains using some of the new concepts developed.

We now briefly describe the contents each of the chapters.

In Chapter II we define the motor as a vector field with a special property, and we then proceed to develop the fundamentals of motor calculus. With the new motor definition, addition of motors is defined simply as pointwise addition of the corresponding vector fields and we show that the derivative of a motor is given by pointwise differentiation of the vectors in the corresponding vector field. It is believed that some of the results obtained for motor dyadics and motor derivatives and integrals are new. The dyad expansion that is given for a motor dyadic will be seen to have novel and important applications in the formulation of the dynamic equations for a rigid body, and it is shown that by considering the set of all motors as a normed space enables limiting operations for motor functions to be treated in a more general and rigorous manner.

In Chapter III we consider various geometrical properties of motors and motor operations. We, in effect, consider some of the connections between motor calculus and screw theory. These geometrical aspects of motor calculus are often useful when motor calculus is applied to rigid body kinematics and dynamics. Aside from using the new motor definition to describe geometric properties, a novel derivation of the so-called circle representation of a two-system is given, which is believed to be simpler than any other found in the literature.

In Chapter IV we consider elementary dynamics using the motor calculus developed in Chapter II. Here, the new motor definition is particularly useful for casting the laws of motion in terms of motors.

In Chapter V we consider elementary kinetostatics using material from both Chapters II and III. We consider in particular the static and velocity/acceleration analysis of an open-loop chain. The static analysis for an open-loop chain constitutes a generalization of that found in Lipkin and Duffy (1982) and the equations given here for velocity/acceleration analysis of an open-loop chain are the same as those given in Featherstone (1984).

In Chapter VI we tie together many of the results of Chapters II-V for application to rigid body dynamics. The new definition of a motor enables a relatively simple construction of the inertia dyadic for a rigid body. Further, the dyad expansion representation of the inertia dyadic given here is novel and for the first time it relates directly to the classical principal screws of inertia due to Ball (1900). Finally, the equations of motion for an open-loop chain are derived. As far as the author is aware, both the derivation and the form of these equations are the simplest to appear in the literature thus far.

CHAPTER II
FUNDAMENTALS OF MOTOR CALCULUS

In this chapter motor calculus is developed using the novel vector field definition of the motor. We assume familiarity with elementary vector calculus, linear algebra, and calculus.

2.1 The Motor

We denote vectors and vector quantities by bold lower case letters (e.g., **v**), and, as is customary in engineering, we shall usually confound these two terms. The association or lack thereof of a vector with a point or line determines whether the vector is considered free or bound. Whether a vector is being considered a free, point, or line vector will often not be made explicit (this is also common in engineering). In fact, the same vector may have different associations in different contexts. The reason for this convention is that the association (or lack thereof) is often clear without making formal algebraic distinctions. For example, suppose that vectors **u** and **v** are associated with two distinct points. Then, we assume that such an expression as **u** = **v** implies that **u** and **v** have the same magnitude and direction. Or, the sum **u** + **v** is obtained by assuming **u** and **v** are associated with the same point (i.e., assuming that $\mathbf{u}$ and $\mathbf{v}$ are translated in a parallel sense to the same point so that they can be added), and then this sum may, depending on the context, be a free, point, or line vector.

A vector field is a set of vectors that is indexed by all points in space. A vector at some point P in a vector field $\mathbf{v}$ is denoted $\mathbf{v}_P$. Here, $\mathbf{v}$ can be interpreted as an arbitrary vector in the vector field to which it belongs. We now define the motor as a special kind of vector field and give some associated definitions and properties.

DEFINITION. A motor is a vector field $\mathbf{m}$ for which there is a vector $\tilde{\mathbf{m}}$ such that for every pair of points P, Q, we have

$$\mathbf{m}_Q - \mathbf{m}_P = \tilde{\mathbf{m}} \times \overline{PQ} \quad , \tag{2.1}$$

where $\overline{PQ}$ is a vector directed from P to Q.

The vector $\tilde{\mathbf{m}}$ is called the principal vector* of the motor, and, as regards the above definition, this vector is considered free. In mechanics, however, it is often convenient to consider the principal vector bound.

THEOREM 2.1. The principal vector of a motor is unique.

---

* This term is borrowed from Dimentberg (1965) and its meaning here is equivalent to his definition.

PROOF. Suppose $\tilde{\mathbf{m}}$ and $\tilde{\mathbf{m}}'$ are principal vectors of the motor m and let P, Q be chosen as distinct points such that $\overline{PQ} \cdot \tilde{\mathbf{m}} = \overline{PQ} \cdot \tilde{\mathbf{m}}' = 0$ (i.e., $\overline{PQ}$ is perpendicular to $\tilde{\mathbf{m}}$ and $\tilde{\mathbf{m}}'$). From eqn. (2.1), we must have

$$\tilde{\mathbf{m}} \times \overline{PQ} = \tilde{\mathbf{m}}' \times \overline{PQ} \quad ,$$

or, on rearranging,

$$(\tilde{\mathbf{m}} - \tilde{\mathbf{m}}') \times \overline{PQ} = 0 \quad .$$

Cross multiplying both sides with $\overline{PQ}$ and using the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, we obtain

$$(\overline{PQ} \cdot \overline{PQ})(\tilde{\mathbf{m}} - \tilde{\mathbf{m}}') - [\overline{PQ} \cdot (\tilde{\mathbf{m}} - \tilde{\mathbf{m}}')]\overline{PQ} = 0 \quad . \quad (2.2)$$

Since $\overline{PQ} \neq 0$ (P and Q are distinct by assumption) and

$$\overline{PQ} \cdot (\tilde{\mathbf{m}} - \tilde{\mathbf{m}}') = \overline{PQ} \cdot \tilde{\mathbf{m}} - \overline{PQ} \cdot \tilde{\mathbf{m}}'$$

$$= 0 + 0 \quad ,$$

eqn. (2.2) becomes

$$\tilde{\mathbf{m}} - \tilde{\mathbf{m}}' = 0 \quad .$$

Thus $\tilde{m} = \tilde{m}'$. Q.E.D.

We denote motors by bold capital letters (e.g., **M**, **N**) and the set of all motors by $\underline{M}$. The notation **M**:**m** or **m**:**M** indicates the vector field **m** corresponding to **M**. When we write upper case letters for motors (**M**, **N**, **L**, etc.), we shall always assume that vectors in their corresponding vector fields are denoted by corresponding lower case letters (**m**, **n**, **l**, etc.). Thus, if the vector field **v** is a motor, and we wish to denote this motor by **M** (i.e., **M**:**v**), then we have **m** = **v** (where we assume that **m** and **v** are associated with the same point).

Two motors **M**, **N** are said to be equal, **M** = **N**, if and only if **m** = **n** (i.e., the vector fields are equal pointwise). It follows from Theorem 2.1 that $\tilde{m} = \tilde{n}$.

Finally, the zero motor **M** = **0** is defined as the zero vector field **m** = **0**. It follows from eqn. (2.1) that $\tilde{m} = \mathbf{0}$.

2.2 The Bivector Representation

DEFINITION. A bivector corresponding to a motor **M** is the ordered pair of vectors $(\tilde{\mathbf{m}}, \mathbf{m}_P)$ where P can be any point.

THEOREM 2.2. A motor is characterized by any one of its bivectors and any bivector characterizes a unique motor.

PROOF. This follows immediately from eqn. (2.1) and Theorem 2.1. Q.E.D.

By virtue of this theorem, any motor can be represented by a necessarily unique bivector at a given point. We denote the set of all bivectors at a point P by $\underline{B}_P$. Clearly, $\underline{M}$ and $\underline{B}_P$ have the same cardinal number.* We indicate the bivector of a motor $\mathbf{M}$ at a point P by $(\mathbf{M})_P$; thus, $(\mathbf{M})_P = (\tilde{\mathbf{m}}, \mathbf{m}_P)$.

Two bivectors $(\mathbf{M})_P$, $(\mathbf{N})_P$ are said to be equal, $(\mathbf{M})_P = (\mathbf{N})_P$, if and only if $\tilde{\mathbf{m}} = \tilde{\mathbf{n}}$ and $\mathbf{m}_P = \mathbf{n}_P$. It follows that $(\mathbf{M})_P = (\mathbf{N})_P$ if and only if $\mathbf{M} = \mathbf{N}$.

Finally, the zero bivector is defined as $(0,\mathbf{0})$. It follows that $(\mathbf{0})_P = (0,\mathbf{0})$.

2.3 <u>The Velocity Motor and the Wrench</u>

We introduce here two of the three kinds of motors with which we shall be concerned in rigid body mechanics. As mentioned in the introduction, applying motor calculus to rigid body dynamics comprises the study of the algebraic and differential relationships among wrenches, velocity motors and momentum motors. We introduce the velocity motor and wrench here so that we can discuss some of their mathematical properties in subsequent sections of this chapter and in the next chapter. We reserve the introduction of the momentum motor for Chapter IV. (It

---

* That is, there exists a one-one correspondence between $\underline{M}$ and $\underline{B}_P$.

turns out that the definition of the momentum motor is quite similar to that of the wrench.)

DEFINITION. The velocity distribution of one frame relative to another is called a <u>velocity motor</u>.

Let v be the velocity distribution of frame Σ' relative to frame Σ. That $\mathbf{v}$ is in fact a motor is verified by differentiating (with respect to time) a vector joining two fixed points P,Q in Σ'. On the one hand we have

$$\frac{d}{dt}\,\overline{PQ} = \omega \times \overline{PQ} \quad ,$$

where ω is the so-called angular velocity of Σ' relative to Σ, and on the other hand

$$\frac{d}{dt}\,\overline{PQ} = \mathbf{v}_Q - \mathbf{v}_P \quad .$$

Thus,

$$\mathbf{v}_Q - \mathbf{v}_P = \omega \times \overline{PQ} \quad .$$

The velocity motor V:$\mathbf{v}$ has principal vector $\vartheta = \omega$.

DEFINITION. The moment distribution of a force system is called a <u>wrench</u>.*

---

* Ball (1900) also used the term wrench but not in the same sense as here. We consider Ball's definition in Chapter IV.

Let $\mathbf{m}$ be the moment distribution of a force system, composed of forces $\mathbf{f}_1$, $\mathbf{f}_2$, ..., $\mathbf{f}_n$ (these force vectors can be considered point vectors). In order to show that $\mathbf{m}$ is a motor, let P,Q be arbitrary points and let $\mathbf{r}_P^i$, $\mathbf{r}_Q^i$ be vectors directed from P and Q to the point of application of $\mathbf{f}_i$. Then, we have

$$\begin{aligned} \mathbf{m}_Q - \mathbf{m}_P &= \sum \mathbf{r}_Q^i \times \mathbf{f}_i - \sum \mathbf{r}_P^i \times \mathbf{f}_i \\ &= \sum (\mathbf{r}_Q^i - \mathbf{r}_P^i) \times \mathbf{f}_i \\ &= \sum \overline{QP} \times \mathbf{f}_i \\ &= (\sum \mathbf{f}_i) \times \overline{PQ} \quad . \end{aligned}$$

Thus, the wrench $\mathbf{M}:\mathbf{m}$ has principal vector $\tilde{\mathbf{m}} = \sum \mathbf{f}_i$.

Because of the nature of the cross product, the forces can be slid along their lines of action without altering the moment distribution. Thus, if desired, the forces of a force system can be considered line vectors as regards the wrench of this force system.

We, of course, can be more general and say that the moment distribution of any set of point (or line) vectors is a motor. We shall use this fact when we consider the momentum motor in Chapter IV.

2.4 Algebraic Operations

For every pair of motors $\mathbf{M}$, $\mathbf{N}$ and real scalars α, β (or, more concisely, for all $\mathbf{M}, \mathbf{N} \in \underline{M}$ and $\alpha, \beta \in \underline{R}$), we define scalar multiplication and addition/subtraction by

$$\alpha\mathbf{M} \pm \beta\mathbf{N} : \alpha\mathbf{m} \pm \beta\mathbf{n} \quad ;$$

the cross product by

$$\mathbf{M} \times \mathbf{N} : \tilde{\mathbf{m}} \times \mathbf{n} + \mathbf{m} \times \tilde{\mathbf{n}} \quad ;$$

and the reciprocal product by

$$\mathbf{M} \circ \mathbf{N} = \tilde{\mathbf{m}} \cdot \mathbf{n}_P + \mathbf{m}_P \cdot \tilde{\mathbf{n}} \quad ,$$

where P is any point. In addition, we define $-\mathbf{M} : -\mathbf{m}$ (or, equivalently, $-\mathbf{M} = (-1)\mathbf{M}$).

In order to verify that the vector fields $\alpha\mathbf{m} \pm \beta\mathbf{n}$ and $\tilde{\mathbf{m}} \times \mathbf{n} + \mathbf{m} \times \tilde{\mathbf{n}}$ are in fact motors, we must show that they have principal vectors (such that eqn. (2.1) is satisfied). For every pair of points P,Q we have

$$\begin{aligned} (\alpha\mathbf{m}_Q \pm \beta\mathbf{n}_Q) - (\alpha\mathbf{m}_P \pm \beta\mathbf{n}_P) &= \alpha(\mathbf{m}_Q - \mathbf{m}_P) \pm \beta(\mathbf{n}_Q - \mathbf{n}_P) \\ &= \alpha\tilde{\mathbf{m}} \times \overline{PQ} \pm \beta\tilde{\mathbf{n}} \times \overline{PQ} \\ &= (\alpha\tilde{\mathbf{m}} \pm \beta\tilde{\mathbf{n}}) \times \overline{PQ} \end{aligned}$$

and

$$(\tilde{\mathbf{m}} \times \mathbf{n}_Q + \mathbf{m}_Q \times \tilde{\mathbf{n}}) -$$

$$(\tilde{\mathbf{m}} \times \mathbf{n}_P + \mathbf{m}_P \times \tilde{\mathbf{n}}) = \tilde{\mathbf{m}} \times (\mathbf{n}_Q - \mathbf{n}_P) + (\mathbf{m}_Q - \mathbf{m}_P) \times \tilde{\mathbf{n}}$$

$$= \tilde{\mathbf{m}} \times (\tilde{\mathbf{n}} \times \overline{PQ}) + (\tilde{\mathbf{m}} \times \overline{PQ}) \times \tilde{\mathbf{m}}$$

$$= (\tilde{\mathbf{m}} \times \tilde{\mathbf{n}}) \times \overline{PQ} \quad ,$$

which verify that $\alpha\tilde{\mathbf{m}} \pm \beta\tilde{\mathbf{m}}$ and $\tilde{\mathbf{m}} \times \tilde{\mathbf{n}}$ are the principal vectors of $\alpha\mathbf{M} \pm \beta\mathbf{N}$ and $\mathbf{M} \times \mathbf{N}$ respectively.

The reciprocal product $\mathbf{M} \circ \mathbf{N}$ is independent of the choice for P since for every point Q, it can be shown using eqn. (2.1) that

$$\tilde{\mathbf{m}} \cdot \mathbf{n}_Q + \mathbf{m}_Q \cdot \tilde{\mathbf{m}} = \tilde{\mathbf{m}} \cdot \mathbf{n}_P + \mathbf{m}_P \cdot \tilde{\mathbf{n}} \quad .$$

We note that since $\underline{M}$ is closed under scalar multiplication and addition (i.e., $\alpha\mathbf{M} + \beta\mathbf{N} \in \underline{M}$ for all $\mathbf{M}$, $\mathbf{N} \in \underline{M}$ and $\alpha, \beta \in \underline{R}$), $\underline{M}$ is a vector space.*

We now define analogous operations to those above on the set of bivectors $\underline{B}_P$ such that an ismorphism** is

---

* That all the axioms defining a vector space are satisfied is easily verified.

** An isomorphism is a one-one correspondence that preserves significant relations between two sets.

established between $\underline{B}_P$ and $\underline{M}$. For all $\mathbf{M}$, $\mathbf{N}$ ε $\underline{M}$ and α, β ε $\underline{R}$, the operations are defined by

$$\alpha(\tilde{\mathbf{m}}, \mathbf{m}_P) \pm \beta(\tilde{n}, n_P) = (\alpha\tilde{\mathbf{m}} \pm \beta\tilde{n}, \alpha\mathbf{m}_P + \beta n_P) \quad ,$$

$$(\tilde{\mathbf{m}}, \mathbf{m}_P) \times (\tilde{n}, n_P) = (\tilde{\mathbf{m}} \times \tilde{n}, \tilde{\mathbf{m}} \times n_P + \mathbf{m}_P \times \tilde{n}) \quad ,$$

$$(\tilde{\mathbf{m}}, \mathbf{m}_P) \circ (\tilde{n}, n_P) = \tilde{\mathbf{m}} \cdot n_P + \mathbf{m}_P \cdot \tilde{n} \quad .$$

Then, we have an isomorphism for which

$$(\alpha\mathbf{M} \pm \beta\mathbf{N})_P = \alpha(\mathbf{M})_P \pm \beta(\mathbf{N})_P \quad ,$$

$$(\mathbf{M})_P \times (\mathbf{N})_P = (\mathbf{M} \times \mathbf{N})_P \quad ,$$

$$(\mathbf{M})_P \circ (\mathbf{N})_P = \mathbf{M} \circ \mathbf{N} \quad .$$

Note that $\underline{B}_P$ is a vector space and that $(\cdot)_P : \underline{M} \to \underline{B}_P$ is an onto, invertible linear transformation.

Finally, we define $(\cdot)_P^{-1}$ as the inverse of $(\cdot)_P$.

It is interesting to note that with the above definition of motor addition, we can express a wrench as a sum of motors each of which is the moment distribution of a single force. This then would provide an alternative method for verifying that the moment distribution of a force system is a motor. We need only prove that the moment distribution of a single force is a motor. The sum then is a motor since $\underline{M}$ is closed under addition.

2.5 Some Linear Algebra Concepts

2.5.1 Coordinates and Subspaces

THEOREM 2.3. $\dim \underline{M} = \dim \underline{B_P} = 6$.*

PROOF. Since there exists an invertible linear transformation of $\underline{M}$ onto $\underline{B_P}$, namely, $(\cdot)_P$, we have that $\dim \underline{M} = \dim \underline{B_P}$. Now, let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a linearly independent set of vectors. Then, the set $\{(\mathbf{e}_1, \mathbf{0}), (\mathbf{e}_2, \mathbf{0}), (\mathbf{e}_3, \mathbf{0}), (\mathbf{0}, \mathbf{e}_1), (\mathbf{0}, \mathbf{e}_2), (\mathbf{0}, \mathbf{e}_3)\}$ is clearly a basis for $\underline{B_P}$. Thus $\dim \underline{B_P} = 6$. Q.E.D.

An isomorphism (with respect to addition and scalar multiplication) between $\underline{M}$ and $\underline{R}^6$ can be established once a basis is selected for $\underline{M}$. Let $\{\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_6\}$ be a basis for $\underline{M}$. Then the scalars $\alpha_1, \alpha_2, \ldots, \alpha_6$ such that $\mathbf{M} = \sum_{i=1}^{6} \alpha_i \mathbf{M}_i$ are called the <u>coordinates</u> of $\mathbf{M}$ with respect to this basis. We use the common notation $[\mathbf{M}]$ to denote a column vector of coordinates (i.e., a coordinate vector):

---

* $\dim \underline{M}$ and $\dim \underline{B_P}$ refer to the dimensions of these vector spaces; the dimension of a vector space is the number of elements in a basis.

$$[\mathbf{M}] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} .$$

By assuming the standard definitions for addition and scalar multiplication of matrices (here, $[\mathbf{M}]$ is a 6×1 matrix), we establish an isomorphism between $\underline{M}$ and $\underline{R}^6$. Since $(\mathbf{M}_6)_P = \sum_{i=1}^{6} \alpha_i (\mathbf{M}_i)_P$ and $\{(\mathbf{M}_1)_P, (\mathbf{M}_2)_P, \ldots, (\mathbf{M}_6)_P\}$ is a basis for $\underline{B}_P$, we have that $\alpha_1, \alpha_2, \ldots, \alpha_6$ are also the coordinates of $(\mathbf{M})_P$ with respect to this basis of bivectors.

Let $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_6$ be defined such that $(\mathbf{E}_1)_P = (\mathbf{e}_1, \mathbf{0})$, $(\mathbf{E}_2)_P = (\mathbf{e}_2, \mathbf{0})$, $(\mathbf{E}_3)_P = (\mathbf{e}_3, \mathbf{0})$, $(\mathbf{E}_4)_P = (\mathbf{0}, \mathbf{e}_1)$, $(\mathbf{E}_5)_P = (\mathbf{0}, \mathbf{e}_2)$, $(\mathbf{E}_6)_P = (\mathbf{0}, \mathbf{e}_3)$ where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal set of vectors. Then, $\{\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_6\}$ is a basis, which we call a <u>standard basis</u> at point P. Such a basis, it turns out, is often convenient to use and is commonly used in the literature.* When we refer a motor $\mathbf{M}$ to a standard basis we shall often simply write

---

* Featherstone (1984) defined the term standard basis equivalently. We shall show in Chapter III that there is a rectangular set of coordinate axes associated with every standard basis.

$$[\mathbf{M}] = \begin{bmatrix} \tilde{m} \\ m_P \end{bmatrix}$$

instead of writing the six coordinates.

We denote the subspace spanned by any set $\underline{S}$ of motors by $\langle\underline{S}\rangle$. If $\underline{N}$ is an n-dimensional subspace of $\underline{M}$ and $\{M_1, M_2, \ldots, M_n\}$ is a basis for $\underline{N}$, then we have that $\langle\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{N}_n\rangle = \underline{N}$ where for simplicity we have dropped the set brackets. We define the coordinates of a motor $\mathbf{M} \in \underline{N}$ with respect to the basis for $\underline{N}$ in a like manner as above; thus,

$$[\mathbf{M}_n] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

where $\mathbf{M} = \sum_{i=1}^{n} \alpha_i \mathbf{M}_i$. On selecting a basis for $\underline{N}$, an isomorphism is established between $\underline{N}$ and $\underline{R}^n$.

2.5.2 On the Reciprocal Product

The reciprocal product is a special kind of bilinear scalar product on $\underline{M}$. It is particularly useful in rigid body dynamics (in ways analogous to the usefulness of the dot product in particle dynamics). Here, we give some definitions and properties concerning this product.

DEFINITIONS. Two motors, $\mathbf{M}$, $\mathbf{N}$ are said to be reciprocal to each other (or, simply, reciprocal) if $\mathbf{M} \circ \mathbf{N} = 0$. A motor $\mathbf{M}$ is self-reciprocal if $\mathbf{M} \circ \mathbf{M} = 0$.

Unlike the dot product for vectors, the reciprocal product is not positive definite; that is, there exist motors that are self-reciprocal. For example, if $(\mathbf{M})_P = (\tilde{\mathbf{m}}, 0)$ or $(\mathbf{M})_P = (0, \mathbf{m})$, then $\mathbf{M} \circ \mathbf{M} = 0$. The reciprocal product, however, is non-degenerate; that is, when the reciprocal product is expressed in matrix form,

$$\mathbf{M} \circ \mathbf{N} = [\mathbf{M}]^T Q[\mathbf{N}] \quad ,$$

the 6×6 matrix $\mathbf{Q}$ has full rank (or equivalently, is non-singular) for all bases that induce this matrix form. Now, let Q_{ij} denote the i,j^{th} entry of $\mathbf{Q}$. Then,

$$Q_{ij} = \mathbf{M}_i \circ \mathbf{M}_j$$

where $\{\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_6\}$ is a basis to which the coordinates of $[\mathbf{M}]$, $[\mathbf{N}]$ refer. Referring to a standard basis, we have

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

for which $|Q| = -1$ (thus verifying that $\mathbf{Q}$ has full rank). A diagonal form for $\mathbf{Q}$ is obtained by referring to a co-reciprocal basis; that is, a basis for which

$$\mathbf{M}_i \circ \mathbf{M}_j = 0 \qquad \text{for} \qquad i \neq j \quad .$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal set. Then, the set $\{\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_6\}$ such that

$$(\mathbf{M}_1)_P = (\mathbf{e}_1, \mathbf{e}_1), \; (\mathbf{M}_2)_P = (\mathbf{e}_1, -\mathbf{e}_1),$$

$$(\mathbf{M}_3)_P = (\mathbf{e}_2, \mathbf{e}_2), \; (\mathbf{M}_4)_P = (\mathbf{e}_2, -\mathbf{e}_2),$$

$$(\mathbf{M}_5)_P = (\mathbf{e}_3, \mathbf{e}_3), \; (\mathbf{M}_6)_P = (\mathbf{e}_3, -\mathbf{e}_3)$$

is a particular co-reciprocal basis for which

$$Q = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \quad .$$

In this diagonal form, we see that the signature* of **Q** is zero and that, again, **Q** has full rank.

Ball (1900) worked almost exclusively with co-reciprocal bases. (In fact, the terms reciprocal and co-reciprocal are due to Ball.) The screw coordinates that Ball used are equivalent to the coordinates of a motor that is referred to a co-reciprocal basis. Most of the equations in Ball's work are expressed in terms of these coordinates. It must be emphasized that it is considered desirable in this dissertation to obtain equations and expressions in the more general coordinate-free form. A special kind of motor dyadic defined by von Mises (1924a) (which is covered in the next subsection) enables an elegant means of expressing many of Ball's results concerning the dynamics of a rigid body in coordinate-free form. The importance of the reciprocal product and co-reciprocal bases is immediately manifest in this dyadic (as we shall see).

As regards the co-reciprocal basis given above, which was used to obtain the diagonal form for **Q**, we note that each motor in this basis is not self-reciprocal. We can make a general statement concerning this.

---

* The signature of a bilinear form is defined as the number of positive eigenvalues minus the number of negative eigenvalues of its representing matrix (for our case **Q**). The signature, like the rank, is independent of the choice of basis.

THEOREM 2.4. Let the set of non-zero motors $\{\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_6\}$ be co-reciprocal. Then, this set is linearly independent if and only if each motor in the set is not self-reciprocal.

PROOF. We must show that $\alpha_1 = \alpha_2 = \ldots = \alpha_6 = 0$ is the only solution for $\alpha_1\mathbf{M}_1 + \alpha_2\mathbf{M}_2 + \ldots + \alpha_6\mathbf{M}_6 = \mathbf{0}$. Now, for some j,

$$\mathbf{M}_j \circ (\alpha_1\mathbf{M}_1 + \alpha_2\mathbf{M}_2 + \ldots \alpha_6\mathbf{M}_6) = \mathbf{M}_j \circ \mathbf{0} \quad .$$

Or,

$$\alpha_j\mathbf{M}_j \circ \mathbf{M}_j = 0 \quad .$$

If $\mathbf{M}_j$ is not self-reciprocal, then it follows that $\alpha_j = 0$.

Conversely, suppose that $\mathbf{M}_j$ is self-reciprocal. Assume that the set $\{\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_6\}$ is linearly independent. Now, if we choose this basis to obtain the matrix expression for the reciprocal product, that is $[\mathbf{M}]^T\mathbf{Q}[\mathbf{N}]$, we find that Q, which is a diagonal matrix, has a zero diagonal entry corresponding to $\mathbf{M}_j \circ \mathbf{M}_j$. This implies that Q is less than full rank, which contradicts the assumption of linear independence. Q.E.D.

(Note that this theorem can be generalized to any non-degenerate scalar product that is not positive definite.)

DEFINITIONS. Let $\underline{N} \subset \underline{M}$ be a subspace. Then, the <u>reciprocal subspace</u> $\underline{N}^R$ is the set of all motors reciprocal to every motor in $\underline{N}$. The two subspaces $\underline{N}$, $\underline{N}^R$ are said to be <u>reciprocal</u> to each other (or, simply reciprocal). The subspace $\underline{N}$ is said to be <u>self-reciprocal</u> if $\underline{N} \subset \underline{N}^R$.

That $\underline{N}^R$ is a subspace follows from the bilinearity of the reciprocal product.

Since the reciprocal product is a special kind of non-degenerate scalar product, theorems concerning these products in general apply to the reciprocal product inparticular. We state two such theorems whose proofs can be found in (Lang, 1966).*

THEOREM 2.5. Let $\underline{N}^R$ be reciprocal to $\underline{N}$. Then, $\dim \underline{N}^R + \dim \underline{N} = 6$.

THEOREM 2.6. Every subspace of $\underline{M}$ has a co-reciprocal basis.

In Lang, these theorems result from a rather general treatment of non-degenerate scalar products. A less abstract approach can be found in (Sugimoto and Duffy, 1982), where they exploit a one-one correspondence between so-called reciprocal and orthogonal screw systems (these screw systems are what we would call subspaces).

---

* See Lang's Theorem 5 on p. 129 and Theorem 9 on p. 135.

It is important to note that a subspace $\underline{N}$ and its reciprocal subspace $\underline{N}^R$ need not be complements;* this is because of the existence of self-reciprocal motors. Consider, for example, the subspace $\underline{N} = \langle \mathbf{M} \rangle$ where $\mathbf{M}$ is self-reciprocal. It then follows that $\underline{N}$ is self-reciprocal; that is, $\underline{N} \subset \underline{N}^R$. Thus $\underline{N}$ and $\underline{N}^R$ are not complements since $\underline{N} \cap \underline{N}^R = \underline{N} \neq \{0\}$. Finally, note that a self-reciprocal subspace can have dimension no greater than three. For, if $\underline{N}$ is self-reciprocal, then we must have $\dim \underline{N} \leq \dim \underline{N}^R$ (as $\underline{N} \subset \underline{N}^R$), and Theorem 2.5 would be contradicted if $\dim \underline{N} > 3$.

2.5.3 Dyadics, Induced Inner Products and Norms of Symmetric Dyadics

We represent linear transformations on $\underline{M}$ by means of dyadics. Motor dyadics were originally employed by von Mises (1924a and 1924b) and, as we shall see in Chapter VI, are well-suited for use in dynamics.

DEFINITIONS. A dyad $\mathbf{MN}$ comprises two motors and represents a linear transformation such that for every motor $\mathbf{L}$,

$$(\mathbf{MN})\mathbf{L} = \mathbf{M}(\mathbf{N} \circ \mathbf{L}) \quad .$$

A dyadic is a sum of dyads $\mathbf{A} = \sum_{i=1}^{n} \mathbf{M}_i \mathbf{N}_i$ defined by

* $\underline{N}$ and $\underline{N}^R$ would be complements, by definition, if $\underline{N} \cap \underline{N}^R = \{0\}$.

$$\mathbf{A}\mathbf{L} = \sum_{i=1}^{n} \mathbf{M}_i (\mathbf{N}_i \circ \mathbf{L}) \quad .$$

We shall call a dyadic symmetric if it is symmetric with respect to the reciprocal product; that is, for every pair $\mathbf{M}$, $\mathbf{N}$, we have

$$\mathbf{M} \circ \mathbf{A}\mathbf{N} = \mathbf{N} \circ \mathbf{A}\mathbf{M} \quad .$$

(Von Mises defined symmetric dyadics likewise.) We shall be primarily concerned with symmetric dyadics, and we now give several associated definitions and properties of symmetric dyadics.

If the bilinear form $\mathbf{M} \circ \mathbf{A}\mathbf{N}$ of a symmetric dyadic $\mathbf{A}$ is positive definite (i.e., $\mathbf{M} \circ \mathbf{A}\mathbf{M} > 0$ if and only if $\mathbf{M} \neq 0$), then $\mathbf{A}$ is said to induce a Euclidean inner product $(\cdot|\cdot)$ defined by $(\mathbf{M}|\mathbf{N}) = \mathbf{M} \circ \mathbf{A}\mathbf{N}$. The induced Euclidean norm $\| \cdot \|$ is defined by $\|\mathbf{M}\| = (\mathbf{M}|\mathbf{M})^{1/2}$. All inner products that we shall consider are Euclidean.

We call the eigenvectors of a dyadic eigenmotors.

THEOREM 2.7. If a symmetric dyadic A induces an inner product $(\cdot|\cdot)$, then there exists a co-reciprocal, linearly independent set of six eigenmotors $\{\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_6\}$. Moreover, with $\lambda_1, \lambda_2, \ldots, \lambda_6$ denoting the corresponding eigenvalues, the dyadic can be expressed in the form

$$\mathbf{A} = \sum_{i=1}^{6} \frac{\lambda_i}{\mathbf{M}_i \circ \mathbf{M}_i} \mathbf{M}_i \mathbf{M}_i \quad . \tag{2.3}$$

PROOF. Since $\mathbf{A}$ is symmetric with respect to the reciprocal product, $\mathbf{A}$ is also symmetric with respect to $(\cdot|\cdot)$; for, $(\mathbf{M}|\mathbf{AN}) = \mathbf{M} \circ (\mathbf{AN}) = (\mathbf{AM}) \circ \mathbf{AN} = (\mathbf{AM}|\mathbf{N})$. Thus, there exists an orthogonal set of six eigenmotors (i.e., the set is orthogonal with respect to $(\cdot|\cdot)$), and the corresponding eigenvalues must be real. In order to show that the set of eigenmotors $\{\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_6\}$ is co-reciprocal, we note that

$$\begin{aligned}(\mathbf{M}_i|\mathbf{M}_j) &= \mathbf{M}_i \circ \mathbf{AM}_j \\ &= \mathbf{M}_i \circ \lambda_j \mathbf{M}_j \\ &= \lambda_j (\mathbf{M}_i \circ \mathbf{M}_j) \quad .\end{aligned}$$

Now, since $(\cdot|\cdot)$ is positive definite, we must have $\lambda_j > 0$. Since $(\mathbf{M}_i|\mathbf{M}_j) = 0$ for $i \neq j$, we have also that $\mathbf{M}_i \circ \mathbf{M}_j = 0$.

In order to prove the latter part of the theorem, we merely verify that the eigenmotors of $\sum_{i=1}^{6} \frac{\lambda_i}{\mathbf{M}_i \circ \mathbf{M}_i} \mathbf{M}_i \mathbf{M}_i$ are the same as those of $\mathbf{A}$:

$$\begin{aligned}\sum_{i=1}^{6} \left(\frac{\lambda_i}{\mathbf{M}_i \circ \mathbf{M}_i} \mathbf{M}_i \mathbf{M}_i\right)\mathbf{M}_j &= \frac{\lambda_j}{\mathbf{M}_j \circ \mathbf{M}_j} \mathbf{M}_j (\mathbf{M}_j \circ \mathbf{M}_j) \\ &= \lambda_j \mathbf{M}_j \quad .\end{aligned}$$

Now, that $\mathbf{M}_i \circ \mathbf{M}_i \neq 0$ can be proven two ways. Since $\{\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_6\}$ is co-reciprocal and linearly independent, by Theorem 2.6 of subsection 2.5.2 we have that the motors cannot be self-reciprocal. Alternatively, $\mathbf{M}_i \circ \mathbf{M}_i = \frac{1}{\lambda_i} (\mathbf{M}_i \circ \lambda_i \mathbf{M}_i) = \frac{1}{\lambda_i} (\mathbf{M}_i \circ \mathbf{A}\mathbf{M}_i) = \frac{1}{\lambda_i} (\mathbf{M}_i | \mathbf{M}_i) > 0$. Q.E.D.

We call the expression $\sum_{i=1}^{6} \frac{\lambda_i}{\mathbf{M}_i \circ \mathbf{M}_i} \mathbf{M}_i \mathbf{M}_i$ a <u>dyad expansion</u> for **A**. This is similar to the spectral decomposition of a linear transformation; if we were to define the dyad $\mathbf{M}_i \mathbf{M}_i$ such that $(\mathbf{M}_i \mathbf{M}_i)\mathbf{N} = \mathbf{M}_i(\mathbf{M}_i | \mathbf{N})$, then the spectral decomposition of **A** would be $\mathbf{A} = \sum_{i=1}^{6} \frac{\lambda_i}{(\mathbf{M}_i | \mathbf{M}_i)} \mathbf{M}_i \mathbf{M}_i$.

It is interesting to note that if **A** is symmetric with respect to $(\cdot|\cdot)$, then **A** always has a spectral decomposition; yet, if **A** is symmetric with respect to the reciprocal product and does not induce an inner product, then **A** might not have a dyad expansion. This is because eigenmotors of **A** can never be self-orthogonal; but it is possible that some eigenmotors be self-reciprocal (consider the dyadic **MM** where **M** is self-reciprocal).*

Theorem 2.7 constitutes a generalization of some results of Featherstone (1984, see p. 57). It is interesting to note that Featherstone obtained the dyad expansion

---

* That **A** induce an inner product is only a necessary condition for **A** to have a dyad expansion. It is not sufficient (consider the dyadic **MM** where **M** is an element of a co-reciprocal basis for $\underline{M}$, all elements of which can be taken as eigenmotors of **MM**).

$$\mathbf{A} = \sum_{i=1}^{6} \frac{(\mathbf{AM}_i)(\mathbf{AM}_i)}{(\mathbf{M}_i|\mathbf{M}_i)} \quad .$$

The dyads, however, in this dyadic can in fact be replaced with the simpler expression in eqn. (2.3).

Finally, because of the form of eqn. (2.3), an expression for the inverse A^{-1} is known immediately. Now, it is easy to show that A^{-1} induces an inner product and that the eigenmotors of A can also be taken as the eigenmotors of A^{-1}. Thus, we have

$$A^{-1} = \sum_{i=1}^{6} \frac{\lambda_i^{-1}}{\mathbf{M}_i \circ \mathbf{M}_i} \mathbf{M}_i \mathbf{M}_i \quad . \tag{2.4}$$

2.6 Motor Differentiation and Integration

The existence of norms on $\underline{M}$ makes $\underline{M}$ a special kind of metric space: a normed space. We assume that limiting operations on $\underline{M}$ are with respect to Euclidean norms, and we assume that limiting operations .for vectors are with respect to the norm that gives the magnitude of a vector (i.e., $|v| = (\mathbf{v} \cdot \mathbf{v})^{1/2}$), which is also Euclidean. We consider motors as functions of real variables only. Thus, with $t, a \in \underline{R}$, $\lim_{t \to a} \mathbf{M}(t) = L$ means that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $0 < |t - a| < \delta$ implies $\| L - \mathbf{M}(t) \| < \varepsilon$. Because normed, finite-dimensional vector spaces are topologically closed,* we have that all limits for motor functions are motors (hence, $\mathbf{L}$ is a motor).** The notation $\lim_{t \to a} \mathbf{m}(t)$ denotes the vector field such that for every fixed point P in the frame of reference, the corresponding vector in this vector field is $\lim_{t \to a} \mathbf{m}_P(t)$. Before we give the relationship between $\lim_{t \to a} \mathbf{M}(t)$ and $\lim_{t \to a} \mathbf{m}(t)$, we first need to consider a few definitions and properties.

DEFINITIONS. A motor $\mathbf{M}(t)$ is <u>fixed</u> in a frame if for every fixed point P in this frame, $\mathbf{m}_P(t)$ is constant. A set of motors is fixed in a frame if each motor in the set is fixed.

We shall assume for the remainder of this section that all motors are referred to the same frame. Thus, if a motor is fixed, it is fixed with respect to this frame.

Let $\{\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_6\}$ be a fixed basis, and let $\alpha_1(t)$, $\alpha_2(t)$, ..., $\alpha_6(t)$ be the coordinates of $\mathbf{M}(t)$. We shall assume the equivalence of <u>convergence in norm</u> and <u>coordinate convergence</u>; that is, $\lim_{t \to a} \mathbf{M}(t) = 0$ if and only if $\lim_{t \to a} \alpha_i(t)$

---

* A set that contains its limit points is said to be topologically closed or, simply, closed.

** See Lemma 54.2 of Voyevodin (1983, p. 173), which states that any finite-dimensional subspace of a normed space is closed.

= 0 for each i.* It follows that, on referring to a fixed standard basis, we have

$$[\lim_{t\to a} \mathbf{M}(t)] = \begin{bmatrix} \lim_{t\to a} \tilde{\mathbf{m}}(t) \\ \\ \lim_{t\to a} m_P(t) \end{bmatrix} .$$

We now give the relationship between $\lim_{t\to a} \mathbf{M}(t)$ and $\lim_{t\to a} \mathbf{m}(t)$. For simplicity, we suppress some notation so that $\lim \mathbf{M} \equiv \lim_{t\to a} \mathbf{M}(t)$, $\lim \mathrm{m} \equiv \lim_{t\to a} \mathbf{m}(t)$, etc.

THEOREM 2.7. For the motor function $\mathbf{M}(t)$, the existence of $\lim \mathbf{M}$ or $\lim \mathbf{m}$ implies the existence of the other and

$$\lim \mathbf{M} : \lim \mathrm{m} \quad ,$$

for which the principal vector is $\lim \tilde{\mathbf{m}}$.

PROOF. Suppose $\lim \mathbf{M}$ exists, and let $\mathbf{M}$ be referred to a standard basis. Then,

$$[\lim \mathbf{M}] = \begin{bmatrix} \lim \tilde{\mathbf{m}} \\ \\ \lim \mathbf{m}_P \end{bmatrix} .$$

Since P is arbitrary, we have

---

* For a general norm, the proof of this is a bit involved. For a Euclidean norm (which is all that we consider), however, the proof is straightforward (Halmos, 1958, p. 175).

$$\lim \mathbf{M} : \lim \mathbf{m},$$

for which the principal vector is $\lim \tilde{\mathbf{m}}$.

Suppose $\lim \mathbf{m}$ exists. For every pair of points P, Q, which we assume are fixed, we have

$$\mathbf{m}_Q - \mathbf{m}_P = \tilde{\mathbf{m}} \times \overline{PQ} \quad .$$

Taking the limit of both sides,

$$\lim m_Q - \lim m_P = \lim(\tilde{\mathbf{m}} \times \overline{PQ}) \quad .$$

Since $\overline{PQ}$ is arbitrary and fixed, we have

$$\lim(\tilde{m} \times \overline{PQ}) = (\lim \tilde{m}) \times \overline{PQ} \quad .$$

Thus $\lim m$ is a motor with principal vector $\lim \tilde{\mathbf{m}}$. Since

$$\begin{bmatrix} \lim \tilde{m} \\ \lim m_P \end{bmatrix} = [\lim \mathbf{M}] \quad ,$$

we have $\lim \mathbf{m} : \lim \mathbf{M}$. Q.E.D.

We use the notation $\frac{\partial \mathbf{m}}{\partial t}$ in order to indicate that differentiation is taken with respect to fixed points; thus $\frac{\partial \mathbf{m}}{\partial t} = \lim\limits_{t \to a} \frac{\mathbf{m}(t) - \mathbf{m}(a)}{t-a}$.

COROLLARY. For the motor function M(t), the existence of either $\frac{d}{dt}\mathbf{M}$ or $\frac{\partial \mathbf{m}}{\partial t}$ implies the existence of the other and

$$\frac{d}{dt}\mathbf{M} : \frac{\partial m}{\partial t} \quad ,$$

for which the principal vector is $\frac{d\tilde{\mathbf{m}}}{dt}$.

PROOF. Apply the theorem to the motor function

$$\frac{\mathbf{M}(t)-\mathbf{M}(a)}{t-a} : \frac{\mathbf{m}(t)-\mathbf{m}(a)}{t-a} \quad . \text{ Q.E.D.}$$

We shall always assume that limits and derivatives of motor functions exist. In addition, we shall assume that motor functions are integrable. With these assumptions, we give the following theorem without proof.

THEOREM 2.8. For the motor function M(t), we have

$$\int_\alpha^\beta \mathbf{M}(\tau)d\tau : \int_\alpha^\beta \mathbf{m}(\tau)d\tau \quad ,$$

for which the principal vector is $\int_\alpha^\beta \tilde{\mathbf{m}}(\tau)d\tau$.

The important results of this section, as regards the remainder of this work, can be summarized as follows:

$$(\frac{d}{dt}\mathbf{M})_P = (\frac{d\tilde{m}}{dt}, \frac{\partial m_P}{\partial t}) \quad ,$$

$$(\int_\alpha^\beta \mathbf{M}d\tau)_P = (\int_\alpha^\beta \tilde{\mathbf{m}}d\tau, \int_\alpha^\beta m_P d\tau) \quad .$$

2.7 Differentiation in a Moving Frame

The relationship between the time derivatives of a motor taken with respect to two frames is analogous to that for a vector. If $\frac{d}{dt}\mathbf{M}$ and $\frac{d'}{dt}\mathbf{M}$ are derivatives taken with respect to Σ and Σ' and **V** the velocity motor of Σ' relative to Σ, then

$$\frac{d}{dt}\,\mathbf{M} = \frac{d'}{dt}\,\mathbf{M} + V \times \mathbf{M} \quad . \tag{2.5}$$

Before we verify this, we need to introduce two operations associated with vector fields: the <u>material derivative</u> $\left.\frac{d\mathbf{m}}{dt}\right|_{\Sigma'}$ and the <u>gradient</u> $\nabla\mathbf{m}$. The notation $\left.\frac{d\mathbf{m}}{dt}\right|_{\Sigma'}$ denotes the derivative with respect to Σ of a vector **m** associated with a fixed point in Σ' (this vector field is not necessarily a motor). The gradient $\nabla\mathbf{m}$ at a point P is a linear operator defined by

$$\lim_{Q\to P} \frac{|\mathbf{m}_Q - \mathbf{m}_P - (\nabla\mathbf{m})\overline{PQ}|}{|\overline{PQ}|} = 0 \tag{2.6}$$

where P can be approached in any direction by Q. We have immediately that $\nabla\mathbf{m} = \tilde{\mathbf{m}}\times$ (where we define the operator $\tilde{\mathbf{m}}\times$ in the obvious way) since **m** is a motor; for, with this substitution in eqn. (2.6), the numerator is zero for all pairs of points P,Q. Thus the gradient $\nabla\mathbf{m}$ is independent of the choice for P and is determined by the principal vector $\tilde{\mathbf{m}}$.*

---

* In vector analysis, the curl of a vector field v, which is denoted by ∇v, is defined by the operator equation

$$(\nabla v)\times = \nabla v + (\nabla v)^T.$$

Now, for the motor **M**, $(\nabla m)^T = -\nabla\mathbf{m}$ since $\nabla\mathbf{m} = \tilde{\mathbf{m}}\times$. Thus, we have that $\nabla\times m = 2\tilde{\mathbf{m}}$.

Now, by the chain rule of differential calculus, we obtain

$$\left.\frac{d\mathbf{m}}{dt}\right|_{\Sigma'} = \frac{\partial \mathbf{m}}{\partial t} + (\nabla\mathbf{m})\mathbf{m} \quad .$$

(Note that $\frac{\partial \mathbf{m}}{\partial t} = \left.\frac{d\mathbf{m}}{dt}\right|_{\Sigma}$.) On rearranging and substituting for $\nabla\mathbf{m}$, we have

$$\left.\frac{\partial \mathbf{m}}{\partial t}\right|_{\Sigma'} = \frac{\partial' \mathbf{m}}{\partial t} + \tilde{\mathbf{v}} \times \tilde{\mathbf{m}} \quad ,$$

which is the vector field form of eqn. (2.5).

2.8 <u>The Acceleration Motor</u>

The <u>acceleration motor</u> of a frame Σ' relative to Σ is defined as the time derivative of the velocity motor $\mathbf{V}$ relating these two frames:

$$\dot{\mathbf{V}} \equiv \frac{d}{dt}\mathbf{V} : \frac{\partial \mathbf{v}}{\partial t} \quad .$$

It is important to note that $\frac{\partial \mathbf{v}}{\partial t}$ is not the acceleration of the point with which it is associated. Or, more precisely, it is not the <u>total acceleration</u>, which is given by the material derivative $\left.\frac{d\mathbf{v}}{dt}\right|_{\Sigma'}$. The acceleration motor $\dot{\mathbf{V}}$ comprises what is called the <u>local acceleration</u> field of Σ' relative to Σ. The relation between the local and total acceleration can be determined from eqn. (2.7) of the previous section:

$$\frac{\partial \mathbf{v}}{\partial t} = \left.\frac{d\mathbf{v}}{dt}\right|_{\Sigma'} + \mathbf{v} \times \tilde{\mathbf{v}} \quad .$$

2.9 Identities

Algebraic Identities:

$$\mathbf{M} \circ \mathbf{N} = \mathbf{N} \circ \mathbf{M} \quad ,$$

$$\mathbf{M} \times \mathbf{N} = -(\mathbf{M} \times \mathbf{N}) \quad ,$$

$$\mathbf{L} \circ (\mathbf{M} + \mathbf{N}) = \mathbf{L} \circ \mathbf{M} + \mathbf{L} \circ \mathbf{N} \quad ,$$

$$\mathbf{L} \times (\mathbf{M} + \mathbf{N}) = \mathbf{L} \times \mathbf{M} + \mathbf{L} \times \mathbf{N} \quad ,$$

$$(\lambda \mathbf{M}) \circ \mathbf{N} = \lambda(\mathbf{M} \circ \mathbf{N}) \quad ,$$

$$(\lambda \mathbf{M}) \times \mathbf{N} = \lambda(\mathbf{M} \times \mathbf{N}) \quad .$$

The algebraic identities can be proven using algebraic properties of the corresponding vector fields. For example,

$$\mathbf{M} \times \mathbf{N} : \tilde{\mathbf{m}} \times \mathbf{n} + \mathbf{m} \times \tilde{\mathbf{n}}$$

$$= -(\mathbf{n} \times \tilde{\mathbf{m}}) - (\mathbf{n} \times \tilde{\mathbf{m}})$$

$$= -(\tilde{\mathbf{n}} \times \mathbf{m} + \mathbf{n} \times \tilde{\mathbf{m}}) : -(\mathbf{N} \times \mathbf{M}) \quad .$$

Differential Identities:

$$\frac{d}{dt}\,(\lambda\mathbf{M}) = \dot{\lambda}\mathbf{M} + \lambda\dot{\mathbf{M}} \quad ,$$

$$\frac{d}{dt}\,(\mathbf{M} + \mathbf{N}) = \dot{\mathbf{M}} + \dot{\mathbf{N}} \quad ,$$

$$\frac{d}{dt}\,(\mathbf{M} \circ \mathbf{N}) + \dot{\mathbf{M}} \circ \mathbf{N} + \mathbf{M} \circ \dot{\mathbf{N}} \quad ,$$

$$\frac{d}{dt}\,(\mathbf{M} \times \mathbf{N}) = \dot{\mathbf{M}} \times \mathbf{N} + \mathbf{M} \times \dot{\mathbf{N}} \quad .$$

The differential identities can be proven using the definition of derivative and the algebraic identities. (Note that we need not resort to the differential properties of the corresponding vector fields; this is because we have defined limits for motors with respect to a metric on $\underline{M}$.) For example,

$$\begin{aligned}
\frac{d}{dt}\,(\mathbf{M} \times \mathbf{N})\bigg|_{t=a} &= \lim_{t\to a} \frac{\mathbf{M}(t)\times\mathbf{N}(t)-\mathbf{M}(a)\times\mathbf{N}(a)}{t-a} \\
&= \lim_{t\to a} \frac{\mathbf{M}(t)\times\mathbf{N}(t)+\mathbf{M}(t)\times\mathbf{N}(a)-\mathbf{M}(t)\times\mathbf{N}(a)-\mathbf{M}(a)\times\mathbf{N}(a)}{t-a} \\
&= \lim_{t\to a} \frac{[\mathbf{M}(t)-\mathbf{M}(a)]\times\mathbf{N}(a)}{t-a} + \lim_{t\to a} \frac{\mathbf{M}(t)\times[\mathbf{N}(t)-\mathbf{N}(a)]}{t-a} \\
&= \dot{\mathbf{M}} \times \mathbf{N} + \mathbf{M} \times \dot{\mathbf{N}} \quad .
\end{aligned}$$

The following additional identity proves very useful in rigid body dynamics (see Chapter VI):

$$\mathbf{L} \circ \mathbf{M} \times \mathbf{N} = \mathbf{L} \times \mathbf{M} \circ \mathbf{N} \quad .$$

CHAPTER III
GEOMETRICAL CONSIDERATIONS

3.1 The Pitch and Central Axis

DEFINITION. The <u>pitch</u> of a non-zero motor $\mathbf{M}$ is given by

$$h = \begin{cases} \dfrac{\tilde{\mathbf{m}} \cdot \mathbf{m}_P}{\tilde{\mathbf{m}} \cdot \tilde{\mathbf{m}}} & \text{if} \quad \tilde{\mathbf{m}} \neq \mathbf{0} \ , \\ \infty & \text{if} \quad \tilde{\mathbf{m}} = \mathbf{0} \ . \end{cases}$$

A non-zero motor is said to be <u>proper</u> if h is finite and <u>improper</u> if h is infinite. The product $\tilde{\mathbf{m}} \cdot \mathbf{m}_P$ is independent of the choice of point P as can be verified from eqn. (2.1). In fact, $\tilde{\mathbf{m}} \cdot \mathbf{m}_P = \frac{1}{2}\, \mathbf{M} \circ \mathbf{M}$ (for all motors); so we can alternatively write $h = \frac{1}{2} \frac{\mathbf{M} \circ \mathbf{M}}{\tilde{\mathbf{m}} \cdot \tilde{\mathbf{m}}}$ for proper motors.

THEOREM 3.1. For every proper motor, there exists a unique line such that $\mathbf{m}_P = h\tilde{\mathbf{m}}$ for every point P on the line.

PROOF. We assert that the equation of the line such that $\mathbf{m}_P = h\tilde{\mathbf{m}}$, P being a point on the line, is given by

$$\mathbf{r} = \frac{\tilde{\mathbf{m}} \times \mathbf{m}_O}{\tilde{\mathbf{m}} \cdot \tilde{\mathbf{m}}} + k\tilde{\mathbf{m}} \quad , \qquad k \in \underline{R} \quad ,$$

where O is an arbitrary point assumed not on the line and $\mathbf{r} = \overline{OP}$.

By definition of a motor, we have

$$\mathbf{m}_P = \mathbf{m}_O + \tilde{\mathbf{m}} \times \mathbf{r} \quad .$$

Substituting for r and simplifying,

$$\mathbf{m}_P = \mathbf{m}_O + \tilde{\mathbf{m}} \times \left(\frac{\tilde{\mathbf{m}}\times\mathbf{m}_O}{\tilde{\mathbf{m}}\cdot\tilde{\mathbf{m}}} + k\tilde{\mathbf{m}}\right)$$

$$= \mathbf{m}_O + (h\tilde{\mathbf{m}} - \mathbf{m}_O + 0)$$

$$= h\tilde{\mathbf{m}} \quad ,$$

which verifies the assertion. Now, $\mathbf{r} = \frac{\tilde{\mathbf{m}}\times\mathbf{m}_O}{\tilde{\mathbf{m}}\cdot\tilde{\mathbf{m}}}$ for $k = 0$. Since points O and P are distinct (by assumption), we have $\frac{\tilde{\mathbf{m}}\times\mathbf{m}_O}{\tilde{\mathbf{m}}\cdot\tilde{\mathbf{m}}} \neq \mathbf{0}$, which in turn implies $\tilde{\mathbf{m}} \times \mathbf{m}_O \neq \mathbf{0}$. Thus, $\mathbf{m}_O \neq h\tilde{\mathbf{m}}$ (for, $\mathbf{m}_O = h\tilde{\mathbf{m}}$ implies $\tilde{\mathbf{m}} \times \mathbf{m}_O = \mathbf{0}$). Thus the given line is unique. Q.E.D.

The line for a proper motor such that $\mathbf{m}_P = h\tilde{\mathbf{m}}$ for points P on the line is called the <u>central axis</u> of the motor.

THEOREM 3.2. Let $\mathbf{M}$ be proper. Then, every non-zero motor in $\langle\mathbf{M}\rangle$ has the same central axis and pitch. Moreover, only motors in $\langle\mathbf{M}\rangle$ have this central axis and pitch combination.

PROOF. Let P be on the central axis of **M**. Then,

$$\mathbf{m}_P = h\tilde{\mathbf{m}} \qquad (3.1)$$

where h is the pitch of **M**. Multiplying by a non-zero scalar λ,

$$\lambda\mathbf{m}_P = h\lambda\tilde{\mathbf{m}} \quad ,$$

and noting that

$$(\lambda\mathbf{M})_P = (\lambda\tilde{\mathbf{m}}, \lambda\mathbf{m}_P) \quad ,$$

we conclude that P is also on the axis of $\lambda\mathbf{M}$ and that the pitch of $\lambda\mathbf{M}$ is also h.

For the latter part of the theorem, suppose motor **N** has the same central axis and pitch as **M**. Then,

$$\mathbf{n}_P = h\tilde{\mathbf{n}} \quad . \qquad (3.2)$$

Since $\tilde{\mathbf{n}}$ and $\tilde{\mathbf{m}}$ are parallel, there exists a scalar β such that $\tilde{\mathbf{n}} = \beta\tilde{\mathbf{m}}$. Thus, from eqns. (3.1) and (3.2), we also have $\mathbf{n}_P = \beta\mathbf{m}_P$. Now,

$$(\mathbf{N})_P = (\tilde{\mathbf{n}}, \mathbf{n}_P)$$

$$= (\beta\tilde{\mathbf{m}},\ \beta\mathbf{m}_P)$$

$$= (\beta\mathbf{M})_P \quad .$$

Therefore, $N = \beta\mathbf{M} \in \langle\mathbf{M}\rangle$. Q.E.D.

Thus, we have that one-dimensional subspaces of proper motors characterize central axis and pitch combinations.

For improper motors, there is no central axis in the sense defined above. We note that an improper motor $\mathbf{M}$ comprises a constant vector field (i.e., for every pair of points P, Q, $\mathbf{m}_P = \mathbf{m}_Q$). Since $\tilde{\mathbf{m}} = O$ and $h = \infty$, there can be no points P such that $\mathbf{m}_P = h\tilde{\mathbf{m}}$. We now resort to projective geometry in order to define the central axis of an improper motor.

Since one-dimensional subspaces of proper motors characterize central axis and pitch combinations, we likewise define the central axis of an improper motors (which by definition has infinite pitch) as some geometric entity characterized by the one-dimensional subspace to which the motor belongs. In projective geometry, this new type of central axis is known as a line at infinity or a non-Euclidean line, as opposed to a Euclidean line, which is a line in Euclidean space.

By Theorem 3.2 and the above definition of the central axis for a improper motor, we can state the following:

THEOREM 3.3. There exists a one-one correspondence between one-dimensional subspaces of $\underline{M}$ and all central axis and pitch combinations.

3.2 On Representing Lines, Rotors and "Vectors"

By definition, we have that for every non-Euclidean line, there exists a motor (which is improper) that has this line as a central axis. We must, however, establish an analogous fact regarding Euclidean lines.

For some arbitrary Euclidean line, let $\mathbf{v}$ be a non-zero vector bound to the line and $\mathbf{r}$ be directed from an arbitrary point in space to any point on the line. Then, the moment distribution $\mathbf{r} \times \mathbf{v}$ is a motor with principal vector $\mathbf{v}$. (If $\mathbf{v}$ is a force, then this motor is a wrench.) Let $\mathbf{M}$:$\mathbf{r} \times \mathbf{v}$. Then, $\tilde{\mathbf{m}} = \mathbf{v}$ and $\mathbf{m} = \mathbf{r} \times \mathbf{v}$. The pitch h of M is clearly zero since $\tilde{m} \cdot m = 0$. Thus, we have that the given line is the central axis of M since for any point P on the line $m_P = h\tilde{\mathbf{m}}$ (=0), and we can now state the following:

THEOREM 3.4. For every line (Euclidean and non-Euclidean), there exists some motor that has this line as a central axis.

We can thus represent every line (Euclidean and non-Euclidean) by a motor. Therefore, we can, in turn, represent a line by motor coordinates or by a bivector. When a motor $\mathbf{M}$ is of zero or infinite pitch, then the coordinates of $\mathbf{M}$ with respect to a standard basis, that is,

$$[\mathbf{M}] = \begin{bmatrix} \tilde{m} \\ m_P \end{bmatrix} \quad ,$$

constitute the well-known Plücker coordinates of the line* that is the central axis of **M**.

DEFINTITION. A rotor is a motor of zero pitch.

The term rotor is adopted from Clifford (1873), which he defined in the sense of a line vector (i.e., a vector bound to a Euclidean line), and the definition we use is equivalent to Clifford's. The discussion preceding Theorem 3.4 shows that there is a unique motor of zero pitch with a given principal vector and central axis (here, we can consider the principal vector bound to the central axis), and it turns out that our definition for addition of rotors (which, of course, is simply addition of motors of zero pitch) is also equivalent to Clifford's.

Rotors can thus be used to represent any Euclidean line and/or line vector. (In particular, rotors can be used to represent forces when forces are interpreted as line vectors.)

Now, what we call an improper motor is what Clifford defined a vector to be. Thus, a "vector" is the set of all

---

* The Plücker coordinates of a line are called homogeneous coordinates. Coordinates of a geometric entity, say, α_1, α_2, ..., α_n are homogeneous when $\lambda\alpha_1$, $\lambda\alpha_2$, ..., $\lambda\alpha_n$ represent the same geometric entity where λ is an arbitrary non-zero real scalar.

vectors indexed to points in space that have the same magnitude and direction (i.e., a constant vector field). Addition of "vectors," as Clifford defined it, is equivalent to addition of improper motors.

As Clifford pointed out, the set of "vectors" is closed under addition whereas the set of rotors is not.* Clifford designated a motor as that which may be a rotor, "vector," or the new entity that can result from the addition of two rotors. This new entity is what we call a motor of non-zero, non-infinite pitch. We shall show in the next section that such a motor can be decomposed into a rotor and an improper motor (i.e., a "vector").

We close this section with a geometric description of a standard basis. Let $\{\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_6\}$ be such a basis for which

$$(\mathbf{E}_1)_O = (\mathbf{e}_1, \mathbf{0}),\ (\mathbf{E}_2)_O = (\mathbf{e}_2, \mathbf{0}),\ (\mathbf{E}_3)_O = (\mathbf{e}_3, \mathbf{0}) \quad ,$$

$$(\mathbf{E}_4)_O = (\mathbf{0}, \mathbf{e}_1),\ (\mathbf{E}_5)_O = (\mathbf{0}, \mathbf{e}_2),\ (\mathbf{E}_6)_O = (\mathbf{0}, \mathbf{e}_3) \quad .$$

We note that $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ are rotors and $\mathbf{E}_4$, $\mathbf{E}_5$, $\mathbf{E}_6$ are improper motors. The central axes of $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ constitute

---

* We tacitly assume here that the zero "vector" and zero rotor are defined; these would both be the same as the zero motor. Note, however, that we shall maintain the convention that the zero motor is neither proper nor improper (or, equivalently, the pitch is undefined).

a rectangular set of lines that intersect at O (see Fig. 3.1). If we designate the directions of these lines by e_1, e_2, e_3, we then have a Cartesian coordinate system with origin O. Clearly, the first three coordinates of a motor $\mathbf{M}$ with respect to this standard basis are the vector coordinates of $\tilde{m}$ in this Cartesian system and the last three coordinates are the vector coordinates of m_O in this same Cartesian system.

3.3 A Useful Decomposition

Let $\mathbf{M}$ be proper with pitch h. We define

$$\mathbf{M}^0 : \mathbf{m} - h\tilde{\mathbf{m}} \qquad \text{and} \qquad \mathbf{M}^\infty : \tilde{\mathbf{m}} \quad ,$$

so that we have

$$\mathbf{M} = \mathbf{M}^0 + h\mathbf{M}^\infty \quad . \tag{3.3}$$

The superscripts indicate the pitches. We can show simultaneously that the pitch of $\mathbf{M}^0$ is in fact zero and that the central axis of $\mathbf{M}^0$ is the same as that of $\mathbf{M}$: If P is a point on the central axis of $\mathbf{M}$, then

$$\mathbf{m}_P^0 = \mathbf{m}_P - h\tilde{\mathbf{m}} = 0\tilde{m} \quad ,$$

where $\mathbf{m}^0 : \mathbf{M}^0$.

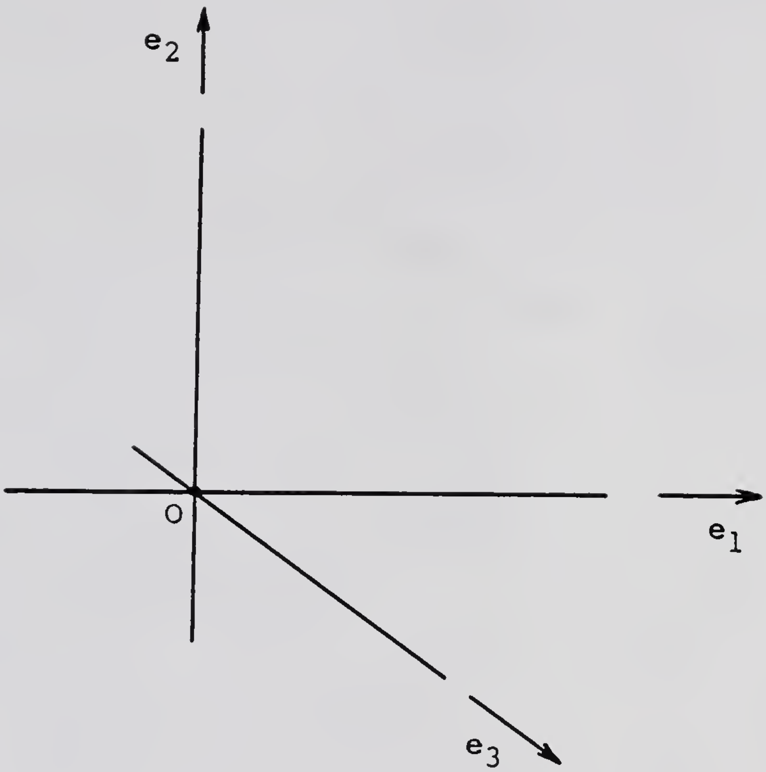


Fig. 3.1. The Cartesian Coordinate System of a Standard Basis

We thus have that every proper motor can be decomposed into a rotor and an improper motor (i.e., a "vector" in the sense of Clifford as mentioned in the preceeding section).

Now, from the previous section we have that

$$\mathbf{M}^0 : \mathbf{r} \times \tilde{\mathbf{m}}$$

where r is directed from an arbitrary point in space to the central axis of $\mathbf{M}^0$. Thus, we have

$$\mathbf{M} : \mathbf{r} \times \tilde{\mathbf{m}} + h\tilde{\mathbf{m}} \quad , \tag{3.4}$$

which enables us to construct a proper motor given the central axis, pitch, and principal vector.

Finally, we note that with respect to a standard basis, $[\mathbf{M}^0]$ comprises the Plücker coordinates of the central axis of **M**.

3.4 A Note on the Dual Operator and Dual Numbers

It is perhaps of interest here to consider a special operator for motors introduced by Clifford (1873), which he designated by ω so defined that

$$\omega\mathbf{M} = \mathbf{M}^\infty \quad \text{and} \quad \omega(\omega\mathbf{M}) = 0$$

for every proper motor **M**. It is easy to show that the rank of the linear transformation represented by ω is three; the null space comprises all improper motors (and the zero motor). (That $\omega\mathbf{N} = \mathbf{0}$ for improper **N** follows from the fact that $\mathrm{N} = \omega((\mathbf{n},\ \mathbf{v}))^{-1}$ where **v** can be any vector and P any point. The motor $((\mathbf{n},\ \mathbf{v}))_P^{-1}$ is, of course, necessarily proper since $\mathbf{n} \neq \mathbf{0}$.) With respect to a standard basis, the matrix of this transformation is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} .$$

We can equivalently define ω as a special kind of number such that for all α, $\beta \in \underline{R}$ and every proper motor **M**,

$$(\alpha + \beta\omega + \omega^2)\mathbf{M} = \alpha\mathbf{M} + \beta\mathbf{M}^{\infty} \quad . \tag{3.5}$$

From this, we can conclude $\omega^2 = 0$ (chose $\alpha = \beta = 0$) and $\omega 0 = 0$ (choose $\beta = 0$). (Note that since $\omega^2\mathbf{M} = \omega(\omega\mathbf{M})$, we have that $\omega\mathbf{N} = \mathbf{0}$ for any improper motor **N**.) Thus, multiplication of a motor by the number ω is equivalent to operating on it by the operator ω.

Numbers of the form $\alpha + \beta\omega$ were called dual numbers by Study (1903) and were applied by him in linear geometry. The term dual operator is often used in recent literature for ω when the operator interpretation is used.

The algebra of dual numbers as induced from eqn. (3.5) is a commutative ring. Some authors who have investigated properties of dual number algebra include Dimentberg (1965), Veldkamp (1976), and Rooney (1975). The motivation for the study of dual number algebra lies primarily in its application to line geometry and rigid body kinematics (Study, 1903; Brand, 1947; Dimentberg, 1965; Yang, 1969, 1974; Yang and Freudenstein, 1964; Veldkamp, 1976; Keler, 1973, 1979). The application of dual number algebra to rigid body dynamics, however, has received much less attention (Dimentberg, 1965; Yang, 1967, 1969, 1971, 1974; Pennock and Yang, 1983).

Dual number algebra is not a particularly useful tool for rigid body dynamics. Although equations of motion can be expressed in dual number form (see the cited works of Dimentberg and Yang), such a form does not, in my opinion, provide a catalyst for explication of dynamic properties. It is for this reason that we will not exploit dual number algebra.

3.5 Streamlines of a Motor

By use of the decomposition in eqn. (3.3), we can easily describe the geometric nature of the vector field of a motor. Let $\mathbf{M}$ be proper and let $\mathbf{r}$ = OP where O is any point and P is on the central axis of $\mathbf{M}$. Then we have $\mathbf{m}_O^0 = \mathbf{r} \times \tilde{\mathbf{m}}$ (where $\mathbf{m}^0{:}\mathbf{M}^0$). The streamlines of $\mathbf{m}^0$ comprise concentric circles about the central axis and in planes normal to the central axis. One such circle is illustrated in Fig. 3.2. By adding $h\tilde{\mathbf{m}}$ to $\mathbf{m}^0$ (note $h\tilde{\mathbf{m}} + \mathbf{m}^0{:}\mathbf{M}$), the streamlines for m become concentric helices of pitch h about the central axis. One such helix is illustrated in Fig. 3.3. The streamlines of $\mathbf{m}^\infty (= \tilde{\mathbf{m}})$ are simply straight lines parallel to the central axis.

3.6 The Unit Motor

We reserve the symbol $\mathbf{S}$ for unit motors. A unit motor S is defined such that $|\tilde{\mathbf{s}}| = 1$ ($|\mathbf{s}| = 1$) if S is proper (improper).

It is important to note that a unit motor is not defined with respect to a norm. The reason for so defining unit motors is that they are convenient for geometric purposes and for use in mechanics. (Von Mises defined unit motors similarly.)

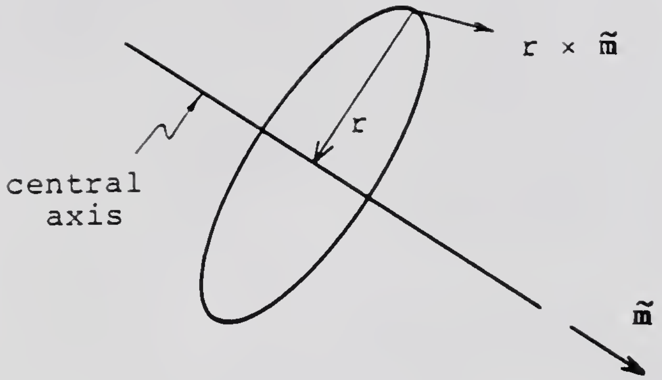


Fig. 3.2. Streamline of a Rotor

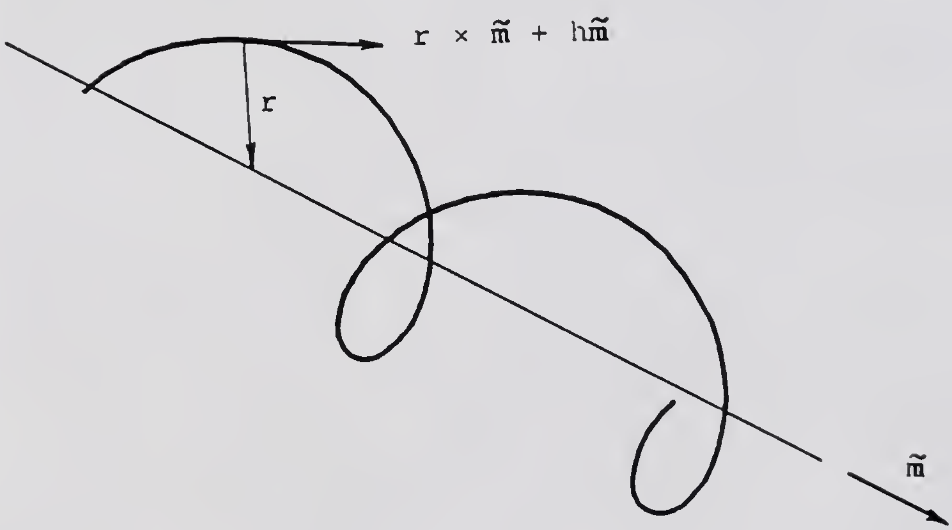


Fig. 3.3. Streamline of a Motor ($h \neq 0$ and $h \neq \infty$)

3.7 The Screw

DEFINITION (Synthetic). A screw is a central axis and pitch combination of some motor.

DEFINITION (Analytic). A screw is a one-dimensional subspace of $\underline{M}$.

The terms synthetic and analytic are taken from projective geometry. The synthetic definition of the screw is the same as that given by Ball (1900). We shall, however, usually confound the two definitions by exploiting Theorem 3.3. The synthetic definition is convenient for visualization purposes, and the analytic definition is convenient for algebraic purposes.

In analogy with motors, we say that a screw is proper (improper) if its pitch is finite (infinite).

We can now define a proper motor in terms of a screw. It is in this manner that Clifford defined the motor: "Just as a vector . . . is magnitude associated with direction, and as a rotor . . . is magnitude associated with an axis; so this new quantity, which is the sum of two or more rotors . . . , is magnitude associated with a screw." It is in fact implicit in Clifford's definition of motor (and rotor) that a sense is assumed for the central axis. Thus, the central axis, pitch, and the magnitude and sense (with respect to the central axis) of the principal vector determines a proper motor.

We can now give Ball's definitions of twist and wrench. A "twist about a screw" is equivalent to a velocity motor. The twist is the magnitude of the angular velocity (or linear velocity if the velocity motor is improper). A "wrench on a screw" is the same as our definition of wrench. Ball's wrench is the magnitude of the net force of the force system (or the couple of the force system if the representing wrench is improper). (As with Clifford, a sense is assumed for the central axes of screws.)

By the use of unit motors, we can make the above notion of "magnitude associated with a screw" precise. Every motor can be expressed as a scalar multiple of a unit motor: $\mathbf{M} = \lambda \mathbf{S}$ where $\lambda = \pm|\tilde{\mathbf{m}}|$ ($\lambda = \pm|\mathbf{m}|$ for improper $\mathbf{M}$) and $\mathbf{S} = \frac{\mathbf{m}}{\lambda}$ ($\mathbf{S}$ can be chosen as any unit motor if $\lambda = 0$). Thus, $\mathbf{S}$ represents the screw of $\mathbf{M}$ together with a direction, where we assume the direction of $\mathbf{S}$ is given by $\tilde{\mathbf{s}}$ ($\mathbf{s}$) if $\mathbf{S}$ is proper (improper). We assume there is no definite direction if $\lambda = 0$. We call λ the <u>magnitude</u> of the motor, which, obviously, may be positive or negative (we note in particular that the magnitude of each unit motor $\mathbf{S}$ may be either ± 1). If $\mathbf{M}$ is a wrench, then λ is the magnitude of a force (torque) if $\mathbf{M}$ is proper (improper); if $\mathbf{M}$ is a velocity motor, then λ is the magnitude of an angular (linear) velocity if $\mathbf{M}$ is proper (improper).

The following theorem is useful in kinematics.

THEOREM 3.5. A unit motor $\mathbf{S}$ is fixed in a frame Σ if and only if

(i) the central axis of $\mathbf{S}$ is fixed in Σ;

(ii) the pitch of $\mathbf{S}$ is constant;

(iii) the direction of $\mathbf{S}$ is constant.

PROOF. The proof is immediate for improper $\mathbf{S}$. If $\mathbf{S}$ is proper, then conditions (i) and (iii) determine $\mathbf{S}^0$, and $\mathbf{S}^0$, which in turn, determines $\mathbf{S}^\infty$. Both $\mathbf{S}^0$ and $\mathbf{S}^\infty$ must be fixed. That the pitch h is constant implies that $\mathbf{S} = \mathbf{S}^0 + h\mathbf{S}^\infty$ is fixed.

The proof of the converse is immediate. Q.E.D.

DEFINITIONS. The screws $\langle \mathbf{M} \rangle$, $\langle \mathbf{N} \rangle$ are said to be *reciprocal* to each other (or, simply, reciprocal) if $\mathbf{M} \circ \mathbf{N} = 0$ (i.e., the motors $\mathbf{M}$, $\mathbf{N}$ are reciprocal). A *motor system* is a subspace of $\underline{M}$. A *screw system* is the set of screws corresponding to a motor system. The *order* of a motor system and its corresponding screw system is the dimension of the motor system. A one-, two-, ..., n-system is a screw system of order one, two, ..., n.

The term screw system is due to Ball and the term motor system is due to Everett (1875).

Motor systems are convenient for characterizing permissible relative motion between rigid bodies in constrained rigid body systems (see Chapter V).

3.8 Geometry of Motor Operations

In this section, we present some geometric properties associated with the reciprocal product, cross product, sum, and linear combination. All the results obtained for the first three operations are well-known. The derivation of the circle representation for a two-system in subsection 3.8.4, however, is believed to be simpler than any other derivations found in the literature.

In the following, we assume that the pitches of the motors $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}$ (and the unit motors S_1, S_2, and S) are h_1, h_2, and h.

3.8.1 The Reciprocal Product

Let $\mathbf{M}_1 = \lambda_1 S_1$ and $\mathbf{M}_2 = \lambda_2 S_2$ be proper, the central axes of which are illustrated in Fig. 3.4. The common normal line intersects the central axes at points P_1 and P_2. We specify the relative position of one axis relative to the other by the vector $d\mathbf{e} = \overline{P_1 P_2}$, where $|\mathbf{e}| = 1$ and $d > 0$, and by the angle α, which is subtended by $\tilde{s}_1$ and $\tilde{s}_2$ and is measured in a right hand sense about **e**.

Consider

$$\mathbf{M}_1 \circ \mathbf{M}_2 = \lambda_1 \lambda_2 S_1 \circ S_2$$

$$= \lambda_1 \lambda_2 (S_1^0 + h_1 S_1^\infty) \circ (S_2^0 + h_2 S_2^\infty)$$

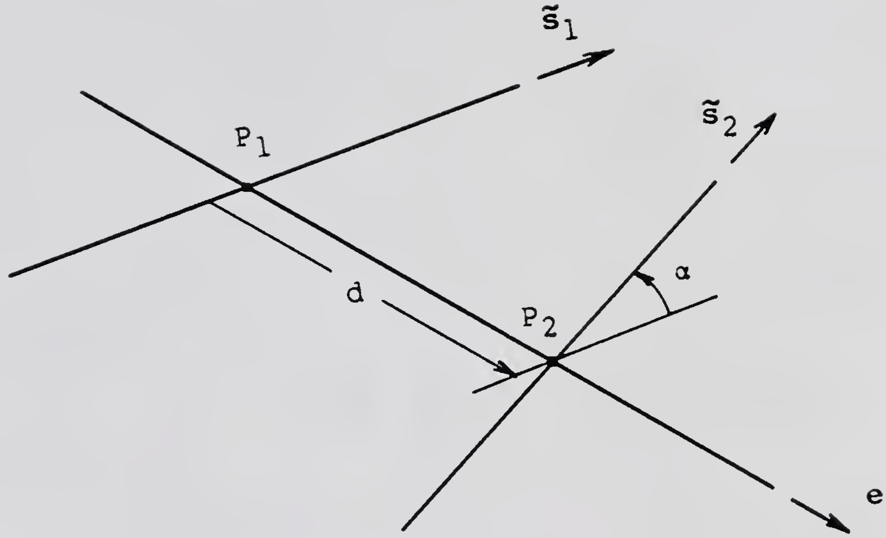


Fig. 3.4. Central Axes of $\mathbf{s}_1$ and $\mathbf{s}_2$

$$= \lambda_1\lambda_2(\mathbf{s}_1^0 \circ \mathbf{s}_2^0 + h_1\mathbf{s}_1^\infty \circ \mathbf{s}_2^0 + h_2\mathbf{s}_2^\infty \circ \mathbf{s}_1^0 + h_1h_2\mathbf{s}_2^\infty \circ \mathbf{s}_2^\infty \quad .$$

Now,

$$\mathbf{s}_1^0 \circ \mathbf{s}_2^0 = (\mathbf{s}_1^0)_{P_1} \circ (\mathbf{s}_2^0)_{P_1}$$

$$= (\tilde{\mathbf{s}}_1, 0) \circ (\tilde{\mathbf{s}}_2, d\mathbf{e} \times \tilde{\mathbf{s}}_2)$$

$$= -d\mathbf{e} \cdot \tilde{\mathbf{s}}_1 \times \tilde{\mathbf{s}}_2$$

$$= -d\mathbf{e} \cdot \mathbf{e}\sin\alpha_{12}$$

$$= -d\sin\alpha_{12} \quad ,$$

$$\mathbf{s}_1^\infty \circ \mathbf{s}_2^0 = \tilde{\mathbf{s}}_1 \cdot \tilde{\mathbf{s}}_2$$

$$= \cos\alpha_{12} \quad ,$$

$$\mathbf{s}_2^\infty \circ \mathbf{s}_1^\infty = \tilde{\mathbf{s}}_2 \cdot \tilde{\mathbf{s}}_1$$

$$= \cos\alpha_{12} \quad ,$$

$$\mathbf{s}_1^\infty \circ \mathbf{s}_2^\infty = 0 \quad .$$

Substituting, we obtain

$$\mathbf{M}_1 \circ \mathbf{M}_2 = \lambda_1\lambda_2[(h_1 + h_2)\cos\alpha - d\sin\alpha] \quad , \qquad (3.6)$$

which is the well-known expression for the reciprocal product of two proper motors.*

Using eqn. (3.6), we can deduce the following:

THEOREM 3.6. Suppose the proper motors $\mathbf{M}_1$, $\mathbf{M}_2$ (screws $\langle\mathbf{M}_1\rangle$, $\langle\mathbf{M}_2\rangle$) have intersecting central axes. Then, the motors (screws) are reciprocal if and only if one or both of the following hold:

(i) $h_1 = -h_2$,

(ii) the central axes are perpendicular.

It is clear that every pair of improper motors (screws) is reciprocal. If $\mathbf{M}_1$ is proper and $\mathbf{M}_2$ improper, then it is also clear that $\mathbf{M}_1 \circ \mathbf{M}_2 = 0$ if and only if $\tilde{\mathbf{m}}_1 \cdot \mathbf{m}_2 = 0$.

3.8.2 The Cross Product

THEOREM 3.7. Let $\mathbf{M}_1$, $\mathbf{M}_2$ be proper motors with non-parallel central axes. Then, $\mathbf{M}_1 \times \mathbf{M}_2$ is a proper motor

---

* Ball called the expression

$$\frac{1}{2}[(h_1 + h_2)\cos\alpha - d\sin\alpha]$$

the virtual coefficient of the screws $\langle S_1\rangle$ and $\langle S_2\rangle$.

whose central axis is the common normal of the central axes of $\mathbf{M}_1$ and $\mathbf{M}_2$.

PROOF. The product $\mathbf{M}_1 \times \mathbf{M}_2$ is proper since $\tilde{\mathbf{m}}_1 \times \tilde{\mathbf{m}}_2 \neq 0$. Now, since

$$\tilde{\mathbf{m}}_1 \times \tilde{\mathbf{m}}_2 \cdot \tilde{\mathbf{m}}_1 = \tilde{\mathbf{m}}_1 \times \tilde{\mathbf{m}}_2 \cdot \tilde{\mathbf{m}}_2 = 0$$

and

$$\mathbf{M}_1 \times \mathbf{M}_2 \circ \mathbf{M}_1 = \mathbf{M}_1 \times \mathbf{M}_2 \circ \mathbf{M}_2 = 0 \quad ,$$

we have by Theorem 3.6 that the central axis of $\mathbf{M}_1 \times \mathbf{M}_2$ is the common normal to the central axes of $\mathbf{M}_1$ and $\mathbf{M}_2$. Q.E.D.

THEOREM 3.8. The product $\mathbf{M}_1 \times \mathbf{M}_2$ is improper if and only if either condition holds:

(i) $\mathbf{M}_1$ is improper and $\mathbf{m}_1 \times \tilde{\mathbf{m}}_2 \neq 0$,

(ii) $\mathbf{M}_1$ and $\mathbf{M}_2$ are proper motors whose central axes are distinct and parallel.

PROOF. The proof is immediate. Q.E.D.

3.8.3 The Motor Sum

THEOREM 3.9. If the proper motors $\mathbf{M}_1$, $\mathbf{M}_2$ have non-parallel central axes (or parallel central axes such that $\tilde{\mathbf{m}}_1 \neq -\tilde{\mathbf{m}}_2$), then the sum $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$ is a proper motor whose central axis intersects the common normal line (or every common normal line) of the central axes of $\mathbf{M}_1$ and $\mathbf{M}_2$.

PROOF. Let the common normal line (or some common normal line) be represented by the rotor $\mathbf{S}$. Then, we must have

$$\tilde{s} \cdot \tilde{m}_1 = \tilde{s} \cdot m_2 = 0 \quad ,$$

and by Theorem 3.6, we have

$$S \circ \mathbf{M}_1 = S \circ \mathbf{M}_2 = 0 \quad .$$

Now, since $\mathbf{m}_1 + \mathbf{m}_2 \neq \mathbf{0}$ (thus $\mathbf{M}$ is proper), we have that the central axis of $\mathbf{M}$ is perpendicular to central axis of $\mathbf{S}$; for,

$$\begin{aligned} \tilde{s} \cdot (\tilde{m}_1 + \tilde{m}_2) &= \tilde{s} \cdot \tilde{m}_1 + \tilde{s} \cdot \tilde{m}_2 \\ &= 0 \quad . \end{aligned}$$

In addition, since

$$\begin{aligned} S \circ (\mathbf{M}_1 + \mathbf{M}_2) &= S \circ \mathbf{M}_1 + S \circ \mathbf{M}_2 \\ &= 0 \quad , \end{aligned}$$

it follows from Theorem 3.6 that the central axis of $\mathbf{M}$ also intersects the central axis of S. Q.E.D.

THEOREM 3.10. Let $\mathbf{M}_1$, $\mathbf{M}_2$ be proper motors whose central axes are parallel and $\mathbf{m}_1 = -\mathbf{m}_2$. Then, the sum $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$ is

(i) improper if the central axes of $\mathbf{M}_1$ and $\mathbf{M}_2$ are distinct or $h_1 \neq h_2$, or, equivalently, $\langle\mathbf{M}_1\rangle \neq \langle\mathbf{M}_2\rangle$;

(ii) zero if the central axes of $\mathbf{M}_1$ and $\mathbf{M}_2$ are the same and $h_1 = h_2$, or, equivalently, $\langle\mathbf{M}_1\rangle = \langle\mathbf{M}_2\rangle$.

PROOF. For both (i) and (ii) we have $\tilde{\mathbf{m}} = \mathbf{m}_1 + \mathbf{m}_2 = \mathbf{0}$. For (i), $\langle\mathbf{M}_1\rangle \neq \langle\mathbf{M}_2\rangle$ implies that $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 \neq \mathbf{0}$; thus, $\mathbf{M}$ is improper. For (ii), $\langle\mathbf{M}_1\rangle = \langle\mathbf{M}_2\rangle$ and $\mathbf{m}_1 = -\mathbf{m}_2$ implies that $\mathbf{M}_1 + \mathbf{M}_2 = \mathbf{0}$.

THEOREM 3.11. If two proper motors $\mathbf{M}_1$, $\mathbf{M}_2$ have non-parallel, coplanar central axes, then the central axis of the sum $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$ passes through the point of intersection of the central axes of $\mathbf{M}_1$ and $\mathbf{M}_2$ if and only if $h_1 = h_2$. Moreover, $h = h_1 (= h_2)$ where h is the pitch of $\mathbf{M}$, and the central axis of $\mathbf{M}$ lies in the plane of the central axes of $\mathbf{M}_1$ and $\mathbf{M}_2$.

PROOF. Let P be the point of intersection of the central axes of $\mathbf{M}_1$ and $\mathbf{M}_2$. Then, we have

$$(\mathbf{M}_1 + \mathbf{M}_2)_P = (\mathbf{M}_1)_P + (\mathbf{M}_2)_P$$

$$= (\tilde{\mathbf{m}}_1, h_1\tilde{\mathbf{m}}_1) + (\tilde{\mathbf{m}}_2, h_2\tilde{\mathbf{m}}_2)$$

$$= (\tilde{\mathbf{m}}_1 + \tilde{\mathbf{m}}_2, h_1\tilde{\mathbf{m}}_1 + h_2\tilde{\mathbf{m}}_2) \quad .$$

Now, $\mathbf{M} + \mathbf{N}$ is proper since $\mathbf{m}_1 + \mathbf{m}_2 \neq \mathbf{0}$. Point P is on the central axis of $\mathbf{M}$ if and only if $(\tilde{\mathbf{m}}_1 + \tilde{\mathbf{m}}_2) \times (h_1 \tilde{\mathbf{m}}_1 + h_2\tilde{\mathbf{m}}_2) = 0$, in which case $h_1 = h_2$ so that $\tilde{\mathbf{m}}_1 + \tilde{\mathbf{m}}_2 = h_1(\tilde{\mathbf{m}}_1 + \tilde{\mathbf{m}}_1)$ $(= h_2(\tilde{\mathbf{m}}_1 + \tilde{\mathbf{m}}_2))$ and $h = h_1 (= h_2)$.

That the three central axes are coplanar follows from Theorem 3.9. Q.E.D.

3.8.4 The Linear Combination

In this subsection, we study the geometry of the linear combination of two motors; that is, we study the geometry of the two-system. (We are assuming, of course, that the two motors constitute a linearly independent set.)

THEOREM 3.12. Every pair of distinct screws in a two-system determines the two-system.

PROOF. Let $\langle S_1 \rangle$ and $\langle S_2 \rangle$ be distinct screws in a two-system. Then, $\langle S_1 \rangle \cap \langle S_2 \rangle = \{\mathbf{0}\}$ implies that $\{S_1, S_2\}$ is linearly independent and thus forms a basis for the motor system that corresponds to the two-system. Q.E.D.

(Note that this theorem cannot be extended to screw-systems of order greater than two: it is possible that $\langle S_1 \rangle$,

$\langle S_2 \rangle$, ..., $\langle S_n \rangle$ be distinct and that $\{S_1, S_2, \ldots, S_n\}$ is linearly dependent for $n > 2$.)

DEFINITION. The ruled surface comprising the central axes of a two-system is called a cylindroid.

The cylindroid was discovered by Hamilton (1830) and named by Cayley.*

A two-system is characterized by its cylindroid and the distribution of pitch for the lines on the cylindroid.

We classify two-systems according to the following scheme:

CLASS 1. All screws are proper.

CLASS 2. All screws are improper.

CLASS 3. Only one screw is improper.

THEOREM 3.13. The above classification scheme is exhaustive and each class is non-empty.

PROOF. Let $\langle S_1 \rangle$ and $\langle S_2 \rangle$ be distinct screws in a two-system. That this classification scheme is exhaustive follows from the fact that if $\langle S_1 \rangle$ and $\langle S_2 \rangle$ are improper, then all screws in the two-system are improper.

---

* In the footnote on p. 20 of Ball's work, he writes, "The name cylindroid was suggested by Professor Cayley in 1871 in reply to a request I made when in ignorance of the previous work of both Plücker and Battaglini, I began to study this surface."

We must show that each class is non-empty. Clearly, there exist two-systems that belong to Class 2. Now, suppose $<\mathbf{S}_1>$ and $<\mathbf{S}_2>$ are proper. If the central axes of these screws are non-parallel, then $<\mathbf{S}_1>$ and $<\mathbf{S}_2>$ determine a two-system that comprises proper screws (i.e., a Class 1 two-system). If the central axes of $<\mathbf{S}_1>$ and $<\mathbf{S}_2>$ are parallel and $\tilde{s}_1 = -\tilde{s}_2$, then the one improper screw of the Class 2 two-system determined $<\mathbf{S}_1>$ and $<\mathbf{S}_2>$ is $<\mathbf{S}_1 + \mathbf{S}_2>$. Q.E.D.

We note that every pair of distinct screws in a Class 1 two-system must have non-parallel central axes; for, a two-system that has a pair of distinct screws whose central axes are parallel must also have an improper screw (see Theorem 3.10). We also note that the proper screws of a Class 3 two-system must have parallel central axes: the central axes of a proper motor and the sum of this motor and any improper motor are parallel. By Theorem 3.9, we also have that the central axes of the proper screws of a Class 3 two-system are coplanar.

DEFINITION. A two-system is called <u>degenerate</u> if it contains at least one improper screw.

Thus, Class 2 and Class 3 two-systems are degenerate, and Class 1 two-systems are non-degenerate.

The cylindroids of degenerate two-systems are simple geometrically. The central axes of a Class 2 two-system all lie in the so-called plane at infinity.* The central axes of the proper screws of a Class 3 two-system are either all the same or lie in a plane.

The cylindroids of non-degenerate two-systems, however, are much more complex, and we devote the remainder of the subsection to non-degenerate two-systems. We shall ultimately obtain the so-called circle representation of a non-degenerate two-system, which was discovered by Lewis (1880) and used to great advantage by Ball (1900). The circle representation quite elegantly illustrates the geometric properties of non-degenerate two-systems as can be evidenced in Ball's work. Since the circle representation applies only to non-degenerate two-systems (this is the motivation for the above definition), we shall assume henceforth that "two-system" means "non-degenerate two-system" and that cylindroids are of two-systems (i.e., non-degenerate ones).

THEOREM 3.14. All lines on a cylindroid intersect a common normal line.

PROOF. Every pair of lines on a cylindroid has a unique common normal line. It follows from Theorem 3.9 that there exists a unique common normal line for all lines on the cylindroid. Q.E.D.

---

* In projective geometry all lines at infinity and all points at infinity lie in the plane at infinity.

The common normal line of a cylindroid is called the <u>central axis</u> of the cylindroid.

LEMMA. To each line on a cylindroid there exists a corresponding unique line on the cylindroid that is perpendicular to it.

PROOF. Let $\langle S_1 \rangle$ and $\langle S_2 \rangle$ be distinct screws of a two-system. A screw in this two-system whose central axis is perpendicular to the central axis of $\langle S_1 \rangle$ is $\langle S_2 - (\tilde{s} \cdot \tilde{s}_2)\ S_1 \rangle$. This perpendicular central axis is unique; for, otherwise, there would be two parallel central axes, which is impossible. Q.E.D.

If $\langle S_1 \rangle$ and $\langle S_2 \rangle$ are distinct screws of a two-system, every screw in this two-system can be given by $\langle c_\psi S_1 + s_\psi S_2 \rangle$ where $c_\psi \equiv \cos\psi$ and $s_\psi \equiv \sin\psi$ and for which $\psi \varepsilon [0, \pi)$. (This follows since it is the ratio $s_\psi : c_\psi$ that determines the screw, and all possible ratios are given with $\psi \varepsilon [0, \pi)$.)

THEOREM 3.15. If the cylindroid of a two-system does not lie in a plane, then there exists a unique pair of lines on the cylindroid that intersect at right angles.

PROOF. Let $\langle S_1 \rangle$ and $\langle S_2 \rangle$ be chosen from a two-system such that their central axes are perpendicular (that there exists a pair follows from the Lemma). Now, all pairs of screws with perpendicular central axes are given by $\langle c_\psi S_1 + s_\psi S_2 \rangle$ and $\langle -s_\psi S_1 + c_\psi S_2 \rangle$ where $\psi \varepsilon [0, \frac{\pi}{2})$. By Theorem 3.6, the necessary and sufficient condition that such a pair of screws also have perpendicular central axes is

$$(c_\psi S_1 + s_\psi S_2) \circ (-s_\psi S_1 + c_\psi S_2) = 0 \quad .$$

Distributing the product and substituting $2h_1 = S_1 \circ S_1$, $2h_2 = S_2 \circ S_2$, $d = -S_1 \circ S_2$, $s_{2\psi} = 2s_\psi c_\psi$, and $c_{2\psi} = c_\psi^2 - s_\psi^2$ yields

$$(h_2 - h_1)\, s_{2\psi} - dc_{2\psi} = 0 \quad .$$

Now, it is not possible that both $h_2 - h_1 = 0$ and $d = 0$; for, if $h_1 = h_2$ and $d = 0$, then, by Theorem 3.11, all the central axes would lie in a plane. Thus, there is a unique solution with $\psi \in [0, \frac{\pi}{2})$ and therefore a unique pair of screws with central axes that intersect at right angles. Q.E.D.

The two screws on a cylindroid that have intersecting perpendicular axes are called the <u>principal screws</u> of a cylindroid. This pair, of course, is unique for cylindroids that do not lie in a plane. If the cylindroid does lie in a plane, then any pair of perpendicular central axes can be chosen for the principal screws.

We now obtain the equation of the cylindroid and the distribution of pitch for a two-system. Let a right-handed Cartesian coordinate system be located such that the x- and y-axes are the principal axes of the cylindroid of the two-system (thus, the z-axis is the central axis). Let $\langle S_1 \rangle$ and $\langle S_2 \rangle$ be the principal screws where $\tilde{s}_1$ and $\tilde{s}_2$ determine the positive x- and y-axis directions. Any screw <S> in the two-system is given by $\langle S \rangle = \langle c_\psi S_1 + s_\psi S_2 \rangle$ where $\psi \in [0, \pi)$. The position of the central axis of <S> can be specified by ψ, which is the angle subtended by $\tilde{s}$ and $\tilde{s}_1$, and by the distance z, which locates the intersection of the central axis with the z-axis (see Fig. 3.5).

The equation of the cylindroid can be obtained by taking the reciprocal product of **S** with the rotor $S^r = -s_\psi S_1^0 + c_\psi S_2^0$ whose axis is perpendicular to both the z-axis and the central axis of **S** and passes through the origin. Using eqn. (3.6) we have

$$S \circ S^r = (h + 0) \cos \frac{3\pi}{2} - z \sin \frac{3\pi}{2}$$

$$= z \quad .$$

We also have

$$S \circ S^r = (c_\psi S_1 + s_\psi S_2) \circ (-s_\psi S_1^0 + c_\psi S_2^0)$$

$$= -s_\psi c_\psi (h_1 - h_2)$$

where we have substituted $h_1 = S \circ S_1^0$, $h_2 = S_2 \circ S_2^0$, and $0 = S_1 \circ S_2^0 = S_2 \circ S_1^0$. Thus, the equation of the cylindroid is given by

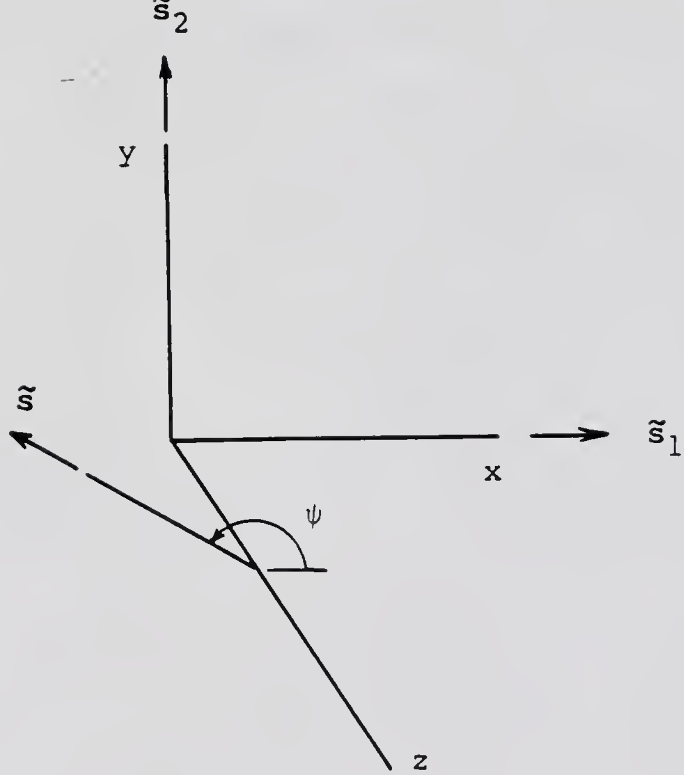


Fig. 3.5. Position of Central Axis on Cylindroid

$$z = -s_\psi c_\psi (h_1 - h_2).$$

The distribution of pitch is given by

$$h = \frac{1}{2}\,\mathbf{S} \circ \mathbf{S}$$

$$= \frac{1}{2}(c_\psi \mathbf{S}_1 + s_\psi \mathbf{S}_2) \circ (c_\psi \mathbf{S}_1 + s_\psi \mathbf{S}_2)$$

$$= c_\psi^2 h_1 + s_\psi^2 h_2$$

where we have substituted $h_1 = \frac{1}{2}\,\mathbf{S}_1 \circ \mathbf{S}_1$, $h_2 = \frac{1}{2}\,\mathbf{S}_2 \circ \mathbf{S}_2$, and $0 = \mathbf{S}_1 \circ \mathbf{S}_2$.

Employing the identities $\frac{1}{2}s_{2\psi} = s_\psi c_\psi$, $\frac{1}{2}(1 + c_{2\psi}) = c_\psi^2$, and $\frac{1}{2}(1 - c_{2\psi}) = s_\psi^2$, the equation of the cylindroid and the distribution of pitch can be expressed as

$$z = -\frac{1}{2}(h_1 - h_2)s_{2\psi} \quad , \tag{3.7}$$

$$h = \frac{1}{2}(h_1 - h_2) + \frac{1}{2}(h_1 - h_2)c_{2\psi} \quad . \tag{3.8}$$

In the hz plane, it is evident that h vs. z is a circle. Assuming without loss of generality that $h_1 > h_2$, a circle representation of the two-system is illustrated in Fig. 3.6. For $h_1 \neq h_2$, each point on the circle represents a unique screw in the two-system; for $h_1 = h_2$, the circle

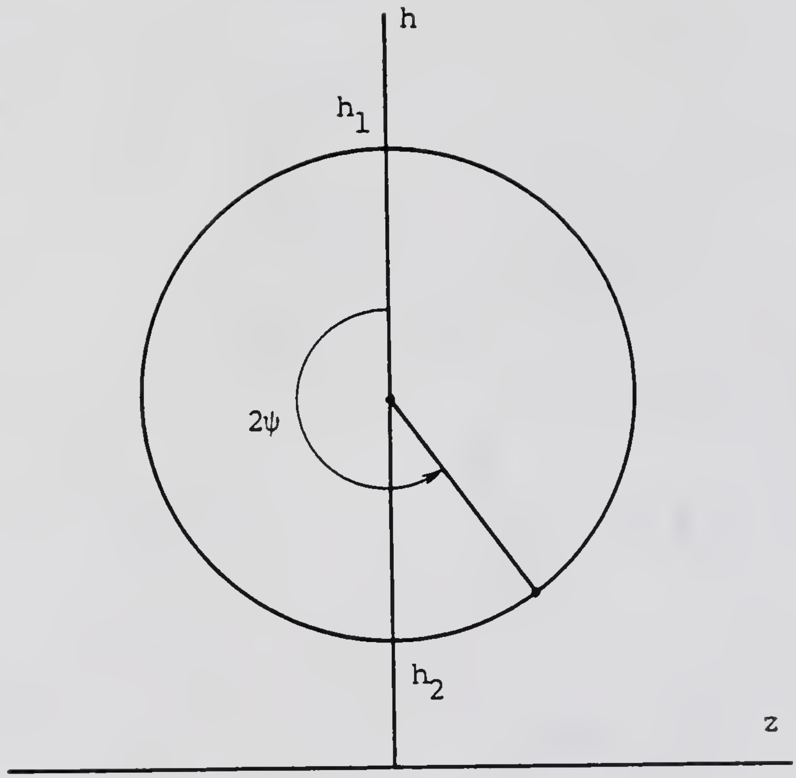


Fig. 3.6. Circle Representation of Two-System

degenerates to a point, which represents all screws in the two-system (the cylindroid lies in a plane). By consideration of the circle representation, the following facts are evident:

(i) The principal screws assume the extreme values of pitch.

(ii) The length of the cylindroid (i.e., along the z-axis) is $|h_1 - h_2|$.

(iii) The plane containing the principal axes (i.e., the x- and y-axes) bisects the length.

CHAPTER IV
ELEMENTARY DYNAMICS

In this chapter we express basic principles of dynamics in terms of motors. We assume familiarity with the basic definitions and principles of analytical mechanics. We only consider finite systems of mass particles (which is the realm of analytical dynamics), because to consider continuous systems would introduce unnecessary complexities. Our primary concern is rigid body mechanics, and it is sufficient for our purposes to consider that a rigid body comprises a finite number of particles.

4.1 The Wrench and Velocity Motor Revisited

In this section we consider some additional definitions and properties associated with the wrench and the velocity motor.

DEFINITIONS. A pure force (rotation) is a wrench (velocity motor) of zero pitch and a couple (translation) is a wrench (velocity motor) of infinite pitch.

We have that any wrench can be expressed as a sum of pure forces, each pure force of which corresponds to a force in the force system of the wrench, or the wrench can be decomposed into a force and couple (see Section 3.3).

Before we can give physical meaning to addition of velocity motors, we need the following

THEOREM 4.1. Let V_{21} be the velocity motor of frame Σ_2 relative to Σ_1 and V_{32} be the velocity motor of frame Σ_3 relative to Σ_2. Then, the velocity motor of Σ_3 relative to Σ_1 is $V_{21} + V_{32}$.

PROOF. This follows immediately from the principle of relative velocities for points. Q.E.D.

Thus any velocity motor can be expressed as a sum of velocity motors, and, in particular, it can be decomposed into a rotation and translation (see Section 3.3).

DEFINITION. Two force systems are said to be <u>equivalent</u> if they have the same wrench.*

All the common reduction procedures for equivalent force systems are easily deduced using motor properties (e.g., that forces are transmissible and that any two forces acting on intersecting lines can be replaced with a single force, the resultant, on a line through the point of intersection).

---

* This is, in fact, equivalent to the standard definition for equivalent force systems (the term equipollent is also sometimes used). In dynamics texts the definition is usually given as follows: two force systems are said to be equivalent if the sum of the forces for each system are the same and the sum of the moments at any point for each system are the same.

4.2 The Momentum Motor

DEFINITION. The angular momentum distribution of a particle system is called a momentum motor.*

Like the wrench, the momentum motor is a moment distribution of a set of vectors, namely, of the linear momentum vectors (or, simply, the linear momenta) of the particle system (we can consider these momenta point vectors). Consider a system of mass particles with masses m_1, m_2, .., m_n located at points P_1, P_2, ..., P_n. Let $\mathbf{r}^i$ be directed from an arbitrary point to P_i. Then, the momentum motor of this system is given by

$$\mathrm{H} : \sum_{i=1}^{n} \mathbf{r}^i \times m_i \mathbf{v}_{P_i} \quad ,$$

where $\mathbf{v}_{P_i}$ is the velocity of P_i. The principal vector is given by

$$\mathbf{\hbar} = \sum_{i=1}^{n} m_i \mathbf{v}_{P_i} \quad ,$$

which, of course, is the linear momentum of the system. Denoting the center of mass by C, the velocity of the center of mass by $\mathbf{v}_C$, and the total mass of the system by m (thus, $m = \sum_{i=1}^{n} m_i$), we have

---

* It appears that Clifford (1878) introduced the momentum motor; he called it the momentum of twist. Dimentberg (1965) erroneously states that Kotelnikov (1895) was the first to define the momentum motor.

$$\tilde{\mathbf{h}} = m\mathbf{v}_C \quad .$$

We say that the momentum motor of a particle system represents the total momentum of the system. A bivector of the momentum motor comprises the linear momentum and angular momentum (at a point) of the system:

$$(\mathbf{H})_P = (m\mathbf{v}_C, \mathbf{h}_P) \quad .$$

4.3 Law of Momentum for a Particle System

DEFINITION. An external (internal) wrench acting on a particle system is the wrench of a force system comprising of external (internal) forces.

We shall assume the strong law of action and reaction: to every force, there is an equal and opposite reaction force with the same line of action.* As is customary in elementary dynamics texts, we assume that this is equivalent to Newton's third law.** In terms of motors, we have that to every pure force, there is an equal and opposite reaction

---

* This is in contrast to the weak law of action and reaction, which states only that forces and their reaction forces be equal and opposite (they need not be colinear).

** According to C. Truesdell, Newton did not actually assert this law. "In Newton's own statement of his third law, there is no explanation of what kinds of 'bodies' he had in mind or what he meant by their actions on each other," Essays in the History of Mechanics, Springer-Verlag, New York, 1968, p. 270.

pure force. Since any wrench can be expressed as a sum of pure forces we can again restate this:

NEWTON'S THIRD LAW. To every wrench, there is an equal and opposite reaction wrench.

For particle systems for which Newton's third law holds, the law of moment of momentum also holds. If H is the momentum motor of a system and **M** the external wrench acting on the system, then this law is given by

$$\mathbf{M} = \dot{\mathrm{H}} \quad , \tag{4.1}$$

where the derivative is taken with respect to an inertial frame (it is, of course, in the vector field form in which this law is usually given: $\mathbf{m} = \frac{\partial \mathbf{h}}{\partial t}$). Now, if **f** denotes the net force of the force system and mv_C the total linear momentum, then we can deduce Newton's second law by simply equating the principal vectors of eqn. (4.1):

$$f = m\dot{\mathbf{v}}_C \quad . \tag{4.2}$$

Newton's second law is also known as the law of linear momentum. Since the law of linear momentum can be deduced from the law of moment of momentum in analytical mechanics (as we have shown here),* we call eqn. (4.1) the law of momentum for a particle system.

Equation (4.1) does not necessarily characterize the dynamics of a particle system; that is, the trajectories of the particles cannot in general be determined given the external wrench as a function of time. We shall not investigate this in detail since it is with rigid bodies that we are primarily concerned, and, as we shall show in Chapter IV, eqn. (4.1) does characterize the dynamics for a rigid body. Suffice it to say, that for an arbitrary particle system, we cannot in general determine the particle velocities at an instant given the particle positions and the momentum motor of the system at that instant. Consider, for example, any two particle systems for which the particles have equal and opposite linear momenta, both vectors of which are parallel to the line on which the particles lie. The momentum motor of all such two particle system is the zero motor.

Of course, eqn. (4.1) can be applied separately to each particle in a particle system, in which case the motion can be completely described. In fact, we need only apply Newton's second law (i.e., eqn. (4.2)) to each particle

---

* For the dynamics of deformable bodies, these two laws are, in general, not dependent.

(note that the velocity of the center of mass of a single particle system is simply the particle velocity).

4.4 Impulse, Conservation of Momentum

On integrating eqn. (4.1) over a time interval $[t_o, t]$, we obtain

$$\int_{t_o}^{t} \mathbf{M} d\tau = \mathbf{H}(t) - \mathbf{H}(t_o) \equiv \Delta\mathbf{H} \tag{4.3}$$

where $\int_{t_o}^{t} \mathbf{M} d\tau$ is the <u>impulse motor</u> imparted to the system. Thus, we have that the impulse motor is equal to the change in the momentum motor $\Delta\mathbf{H}$. If $\mathbf{M} = \mathbf{0}$ then $\mathbf{H}$ will remain constant (i.e., $\Delta\mathbf{H} = \mathbf{0}$); this is the <u>law of conservation of momentum in motor form</u>.

We say that the impulse motor represents the total impulse imparted to a particle system. A bivector of the impulse motor comprises the linear impulse and the angular impulse imparted to the system:

$$\left(\int_{t_o}^{t} \mathbf{M} d\tau \right)_P = \left(\int_{t_o}^{t} \mathbf{f} d\tau, \int_{t_o}^{t} \mathbf{m}_P d\tau \right)$$

where $\mathbf{f}$ is the net force acting on the system. With $(\mathbf{H})_P = (m\mathbf{v}_C, \mathbf{h}_P)$, it follows from eqn. (4.3) that

$$\int_{t_o}^{t} \mathbf{f} d\tau = m\mathbf{v}_C \Big|_{t_o}^{t} ,$$

$$\int_{t_o}^{t} \mathbf{m}_P d\tau = \mathbf{h}_P \Big|_{t_o}^{t} \quad ,$$

from which we can deduce the conservation laws for linear and angular momentum:

$$m\mathbf{v}_C \Big|_{t_o}^{t} = \mathbf{0} \quad ,$$

$$\mathbf{h}_P \Big|_{t_o}^{t} = \mathbf{0}$$

when $\mathbf{f} = \mathbf{m}_P = \mathbf{0}$.

It follows that the total momentum is conserved if and only if both the linear momentum and angular momentum (at any point; hence, all points) is conserved.

Finally, we note that if a constant pure force acts on a particle system, then the angular momentum for points on the central axis of the pure force is conserved, and if a couple acts on the system, then the linear momentum is conserved.

4.5 Power, Work, and Energy for Rigid Bodies

Let a rigid body comprise n mass particles with masses m_i and located at points P_i. Let $\mathbf{V}$ be the velocity motor of the rigid body frame relative to some inertial frame and let $\mathbf{M}$ be the wrench of the force system acting on the body. We assume that the force system comprises n forces f_i (some of which may be zero) acting on the particles. Then, the power

$\underline{\dot{W}}$, which is the time rate of change of the work $\underline{W}$, imparted to the body by the force system is given by

$$\underline{\dot{W}} = \sum \mathbf{f}_i \cdot (\mathbf{v}_O + \tilde{\mathbf{v}} \times \overline{OP}_i)$$

$$= (\sum \mathbf{f}_i) \cdot \mathbf{v}_0 + \tilde{\mathbf{v}} \cdot (\sum \overline{OP}_i \times \mathbf{f}_i)$$

$$= \mathbf{M} \circ \mathbf{V} \quad .$$

We say that $\mathbf{M} \circ \mathbf{V}$ is the power imparted to the body by the wrench $\mathbf{M}$. Now, it can be shown (see any text on elementary dynamics) that for rigid bodies the internal forces impart no power to the body, and that the power imparted to the body (by the external forces) is equal to the time rate of change of kinetic energy $\underline{T}$, which is given by $\underline{T} = \sum \frac{1}{2} m_i \mathbf{v}_{P_i}^2$. So, we have

$$\underline{\dot{W}} = \frac{d}{dt} \sum \frac{1}{2} \mathbf{m}_i \mathbf{v}_{P_i}^2 \quad .$$

Now, let $\mathbf{H}$ be the momentum motor of the system. Again choosing an arbitrary fixed point O in the body frame, we have that

$$\sum m_i \mathbf{v}_{Pi}^2 = \sum (m_i \mathbf{v}_{P_i}) \cdot (\mathbf{v}_O + \tilde{\mathbf{v}} \times \overline{OP}_i)$$

$$= (\sum m_i \mathbf{v}_{P_i}) \cdot \mathbf{v}_O + \tilde{\mathbf{v}} \cdot (\sum OP_i \times (m_i \mathbf{v}_{P_i}))$$

$$= \tilde{\mathbf{h}} \cdot \mathbf{v}_O + \tilde{\mathbf{v}} \cdot \mathbf{h}_O$$

$$= \mathbf{H} \circ \mathbf{V} \quad .$$

Thus, $\underline{T} = \frac{1}{2}\mathbf{H} \circ \mathbf{V}$ is the kinetic energy and

$$\mathbf{M} \circ \mathbf{V} = \frac{d}{dt}\left(\frac{1}{2}\mathbf{H} \circ \mathbf{V}\right) \quad . \tag{4.4}$$

Finally, on integrating eqn. (4.4) over a time interval $[t_o, t]$, we obtain the principle of work and energy:

$$\int_{t_o}^{t} \mathbf{M} \circ \mathbf{V} d\tau = \left(\frac{1}{2}\mathbf{H} \circ \mathbf{V}\right)\Big|_{t_o}^{t} \quad ,$$

where $\int_{t_o}^{t} \mathbf{M} \circ \mathbf{V} d\tau \ \left(= \int_{t_o}^{t} \dot{\underline{W}} d\tau = \underline{W} \Big|_{t_o}^{t} \right)$ is the work imparted over the time interval.

This section should illustrate the central importance of the reciprocal product as regards the dynamics of rigid bodies, as this product is intimately related to the concept of power. The concept of power is important for constrained systems: if the total power imparted to a system by the contraint forces is always zero (i.e., for any possible motion), then there exists a formulation of the equations of motion for the system for which the contraint forces are not present. Such constraints that contribute no power (positive or negative) to the system are known as workless constraints. We shall give the definition of workless constraints in terms of motors in the next chapter.

CHAPTER V
ELEMENTARY KINETOSTATICS

Phillips (1984, p. 4) describes the term kinetostatics as ". . . the study of angular and linear velocities and thus the kinematics of mechanism on the one hand, this being in close conjunction with the study of forces and couples and thus of massless mechanism on the other." It is important to point out that it is instantaneous kinematics to which Phillips is alluding (which is the study of kinematic properties at an instant, that is, the relations of velocities and higher derivatives in mechanisms and machines). The intimate relation between instantaneous kinematics and statics is, perhaps, best illustrated through the use of motor calculus. We consider some elementary aspects of these subjects and, in particular, we consider the instantaneous kinematics and statics of open-loop kinematic chains (or, simply, open-loop chains).

5.1 Constraint Between Bodies

5.1.1 Characterizing Constraint

If a body is constrained to move in a particular manner relative to another body, then we shall assume that, at any instant, the constraint between the bodies is completely characterized by a motor system of permissible relative velocity motors (Ball, 1900, equivalently characterized motion freedom via screw systems). It would be more general to consider arbitrary sets of velocity motors, as opposed to just subspaces; instantaneous kinematic properties would then become much more involved. This, however, is beyond the scope of this work, and it turns out that in many mechanisms and machines (including robots), the constraints can be sufficiently modeled by motor systems.

We shall assume, in addition, that constraints are workless, which we now define in terms of motors.

DEFINITION. The constraint between a body 1 and body 2 is said to be <u>workless</u> if $\mathbf{M} \circ V_{12} = 0$ where $\mathbf{M}$ is the wrench that body 1 exerts on body 2 via the constraint and V_{12} is any velocity motor of body 1 relative to body 2 belonging to the motor system that characterizes the constraint.

We note that $-\mathbf{M}$ is the wrench that body 2 exerts on body 1 (by Newton's third law) and that $V_{21} = -V_{12}$. Since $(-\mathbf{M}) \circ V_{21} = \mathbf{M} \circ V_{12}$, we have that the definition is symmetric.

We call the wrench exerted via a workless constraint a <u>constraint wrench</u>. It is clear that a constraint wrench must belong to a reciprocal motor system. It is thus possible to equivalently characterize a constraint by a reciprocal motor system, which comprises permissible constraint wrenches.

5.1.2 Static Equilibrium

DEFINITION. A body is said to be in static equilibrium if the body's velocity motor with respect to any inertial frame is a constant translation (or the zero motor).

It follows that each particle in the body has a constant velocity so that the net force on each particle is zero. Thus, the net wrench on the body is also zero.

DEFINITION. A body is said to be grounded (or, it is simply called ground) if it is in static equilibrium and maintains static equilibrium regardless of any applied wrench. A body is said to be constrained to ground if its motion is constrained relative to a grounded body.

Consider a rigid body constrained to ground and stationary relative to ground (i.e., the relative velocity motor is zero). Let the constraint be characterized by the motor system $\underline{N}$ and let $\mathbf{M}'$ be an external wrench to be applied and $\mathbf{M}^c$ be the resulting constraint wrench also applied to the body. Since the body is initially stationary relative to ground, it follows that the body is in static eqilibrium before $\mathbf{M}'$ is applied. From the principle of virtual work, we have that the necessary and sufficient condition that the body maintain static equilibrium is

$$\mathbf{M}' \in \underline{N}^R \quad ; \qquad (5.1)$$

for, this is equivilent to saying that $\mathbf{M}'$ imparts no power during a so-called virtual displacement (i.e., $\mathbf{M}' \circ \mathbf{V} = 0$ for all $V \in \mathbf{N}$). Since the net wrench exerted on a body must be zero if the body is in static equilibrium, we can determine the constraint wrench from

$$\mathbf{M}^{C} = -\mathbf{M}' \quad . \tag{5.2}$$

5.2 The Screw Pair

We shall consider the screw pair illustrated in Fig. 5.1 to be the most fundamental joint between two bodies. The screw pair is in fact the most general single degree of freedom lower pair* (Waldron, 1972). Common multiple degree of freedom joints such as the cylindric, ball and socket, planar, and universal can be constructed as special serial combinations of screw pairs (see Duffy, 1980).

As can be seen in the figure, uniform rotation accompanies uniform translation: a rotation of θ about the axis of the screw pair accompanies a translation of $h\theta$ along the axis where h is the pitch of the screw pair. Should $h = \infty$, however, then we define the screw pair to permit only

---

* A lower pair comprises mating surfaces (of the joint between two bodies) that are surfaces of revolution.

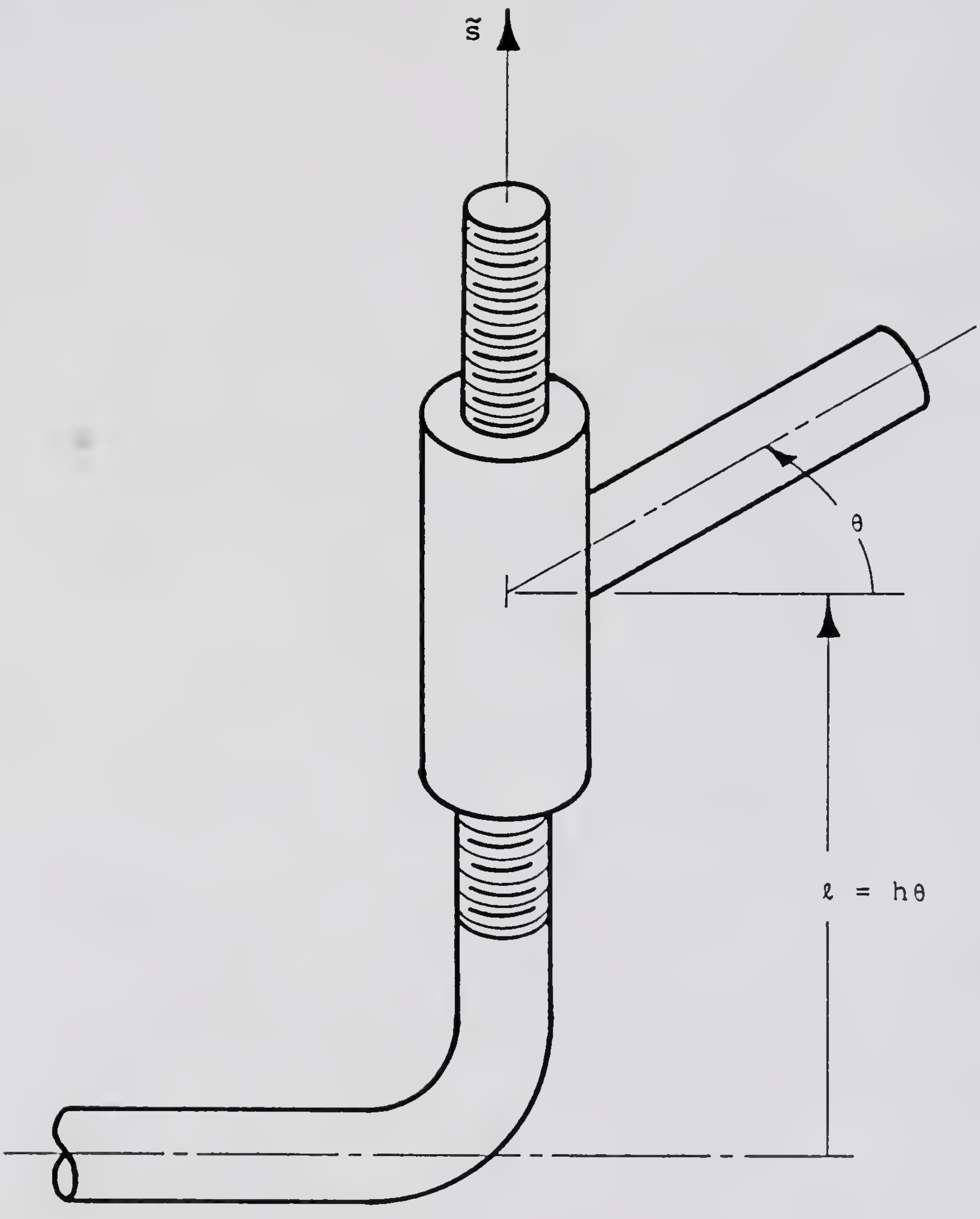
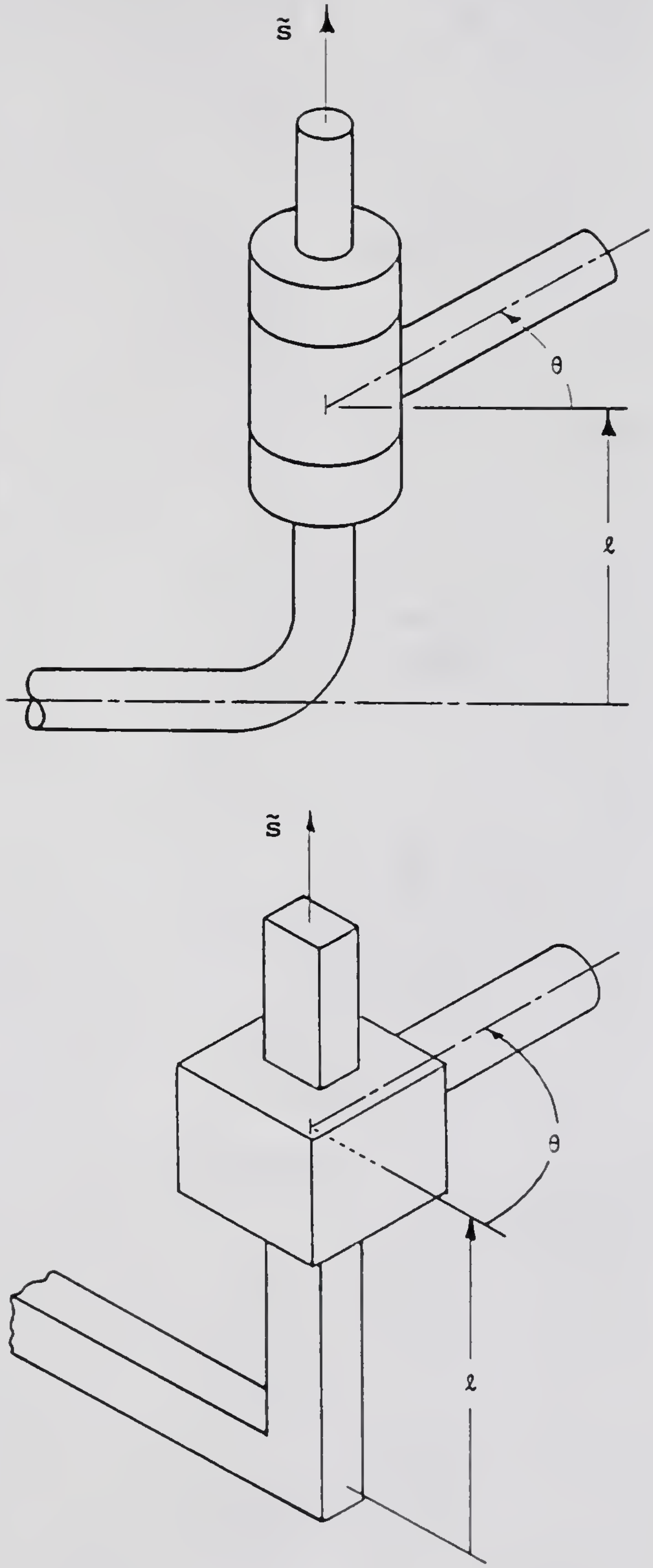


Fig. 5.1. The Screw Pair

translation for which the pair axis can be taken as any line parallel to the translation or the pair axis can be taken as a line at infinity. For $h \neq \infty$, we note that for every value of θ, the screw pair permits a screw (i.e., a one-dimensional subspace) of velocity motors for which the pitch is also h and the central axis is identical to the axis of the screw pair. Since the screw axis is fixed in the two bodies that the screw pair connects, it follows that the screw is fixed in both bodies. Thus, the screw of velocity motors is independent of the value of θ (and is hence the same at every instant). For $h = \infty$, the screw of velocity motors at any instant is clearly improper, which is also fixed in both bodies. If desired, we can take the central axis of this improper screw, which is a line at infinity, as the central axis of the screw pair.

Two practically important special cases of the screw pair are the <u>revolute pair</u> for which $h = 0$ and the <u>prismatic pair</u> for which $h = \infty$, both joints of which are illustrated in Fig. 5.2.

Since a screw pair can be characterized by a screw, it can be characterized by any motor in the screw. In particular, it is very convenient to represent screw pairs by unit motors. Let S represent a screw pair. Then, if the pair is of non-infinite pitch (as in Fig. 5.1), then we assume the direction of S is such that the angle of rotation (θ in Fig. 5.1) is measured in a right sense about $\tilde{s}$; if the



(a)

(b)

Fig. 5.2. The (a) Revolute and (b) Prismatic Pair

pair is prismatic (as in Fig. 5.2b), then we assume the direction of S is such that the length of translation (ℓ in Fig. 5.2b) is measured in a positive sense in the direction of s. The velocity motor can then be given by $\dot{q}S$ where q is the measure of displacement that is, the joint displacement of the screw pair ($q = \theta$ in Fig. 5.1 and $q = \ell$ in Fig. 5.2b).

Finally, we note that by Theorem 3.5, a unit motor that represents a screw pair is fixed in both bodies connected by the pair.

5.3 Open-Loop Chains

5.3.1 Modeling

We shall assume that the joints connecting successive bodies (or links) in an open-loop chain are screw pairs.

Let an open-loop chain comprise n links numbered from the base outward as illustrated in Fig. 5.3. We number the base (or the ground link) 0. The joints are denoted by the unit motors S_i where the i^{th} joint connects links i-1 and i, and the corresponding joint displacement is given by q_i where $q_i = \theta_i$ ($q_i = \ell_i$) if S_i is proper (improper).

5.3.2 Velocity and Acceleration Analysis

Obtaining the equations that are necessary for velocity and acceleration analysis of open-loop chains is straightforward using motor calculus.

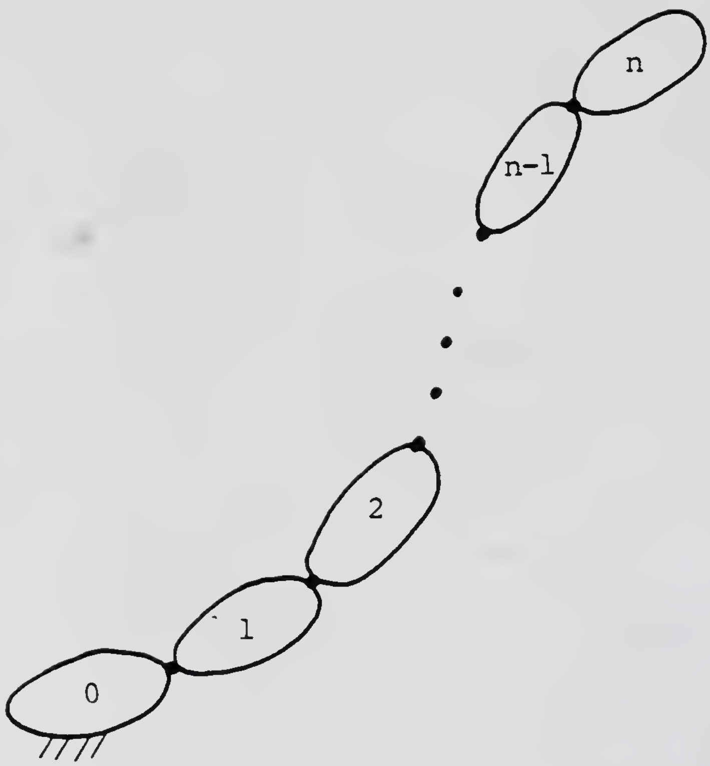


Fig. 5.3. The Open-Loop Chain

Let V_j denote the velocity motor of the j^{th} link relative to the ground link (thus, $\mathbf{V}_0 = \mathbf{0}$). Then, by Theorem 4.1, we have

$$\mathbf{V}_j = \sum_{i=1}^{j} \mathbf{S}_i \dot{q}_i \quad , \tag{5.2}$$

which can be expressed in the recursive form

$$\mathbf{V}_j = \mathbf{V}_{j-1} + \mathbf{S}_j \dot{q}_j \quad . \tag{5.3}$$

Differentiating both sides of eqn. (5.2) (with respect to the ground frame), we obtain

$$\dot{\mathbf{V}}_j = \sum_{i=1}^{j} (\dot{\mathbf{S}}_i \dot{q}_i) \quad .$$

Now, since $\mathbf{S}_i$ is fixed in both links i-1 and i, we have that $\dot{\mathbf{S}}_i = \mathbf{V}_{i-1} \times \mathbf{S}_i = \mathbf{V}_i \times \mathbf{S}_i$ (the time derivatives of $\mathbf{S}_i$ with respect to links i-1 and i are both zero); we choose $\dot{\mathbf{S}}_i = \mathbf{V}_i \times \mathbf{S}_i$ for the following. Thus, we have

$$\dot{\mathbf{V}}_j = \sum_{i=1}^{j} (\mathbf{S}_i \ddot{q}_i + \mathbf{V}_i \times \mathbf{S}_i \dot{q}_i)$$

$$= \sum_{i=1}^{j} [\mathbf{S}_i \ddot{q}_i + (\sum_{k=1}^{i} \mathbf{S}_k \dot{q}_k) \times \mathbf{S}_i \dot{q}_i]$$

$$= \sum_{i=1}^{j} \mathbf{s}_i \ddot{q}_i + \sum_{i=1}^{j} \sum_{k=1}^{i} \mathbf{s}_k \times \dot{q}_i \dot{q}_k \quad . \tag{5.4}$$

Differentiating both sides of eqn. (5.3) we obtain the recursive form

$$\dot{\mathbf{v}}_j = \dot{\mathbf{v}}_{j-1} + \mathbf{s}_j \ddot{q}_j + \mathbf{v}_j \times \mathbf{s}_j \dot{q}_j \quad . \tag{5.5}$$

Summarizing,

$$\mathbf{v}_j = \sum_{i=1}^{j} \mathbf{s}_i \dot{q}_i \quad , \tag{5.2}$$

$$\dot{\mathbf{v}}_j = \sum_{i=1}^{j} \mathbf{s}_i \ddot{q}_i + \sum_{i=1}^{j} \sum_{k=1}^{i} \mathbf{s}_k \times \mathbf{s}_i \dot{q}_i \dot{q}_k \quad , \tag{5.4}$$

which are explicit expressions in the first and second time derivatives of the joint displacements, and

$$\mathbf{v}_j = \mathbf{v}_{j-1} + \mathbf{s}_j \dot{q}_j \quad , \tag{5.3}$$

$$\dot{\mathbf{v}}_j = \dot{\mathbf{v}}_{j-1} + \mathbf{s}_j \ddot{q}_j + \mathbf{v}_j \times \mathbf{s}_j \dot{q}_j \tag{5.5}$$

which are recursive expressions.

5.3.3 Statics

We shall assume that open-loop chains are fully actuated; that is, associated with each joint is an actuator. (Most industrial robots would fall into this category.) Each actuator in a fully actuated open-chain can create a wrench "in-between" the links joined by the associated joint. We now make these notions precise and obtain the equations of equilibrium for the open-loop chain.

We denote the screw of the i^{th} actuator by S_i^a and we assume that this screw is fixed in the i^{th} link. The wrench that the i^{th} actuator exerts on the i^{th} link is given by $\tau_i S_i^a$, and the wrench that is exerted on the $(i-1)^{th}$ link is $-\tau_i S_i^a$ (hence, the actuator creates a wrench "in-between" the two links). We note that τ_i is the magnitude of a force or torque (which we allow to be negative) depending on whether S_i^a is proper or improper.

Suppose that on each link of an n-link open-loop chain an external wrench $\mathbf{M}_i^!$ is exerted and that we require the chain be in static equilibrium; that is, each link be in static equilibrium (we assume, of course, that the base is grounded).

Then, applying condition (5.1) to link i, we obtain

$$\tau_i S_i^a + \sum_{p=i}^{n} \mathbf{M}_p \in \langle S_i \rangle^R \quad ,$$

and thus the constraint wrench $\mathbf{M}_i^c \,\varepsilon\, \langle \mathbf{S}_i \rangle^R$ is determined from

$$\mathbf{M}_i^c = -(\tau_i \mathbf{S}_i^a + \sum_{p=i}^{n} \mathbf{M}'_p) \quad . \tag{5.6}$$

Now, $\mathbf{S}_i \not\varepsilon \langle \mathbf{S}_i \rangle^R$, for, if $\mathbf{S}_i^a \,\varepsilon\, \langle \mathbf{S}_i \rangle^r$ and $\sum_{p=i}^{n} \mathbf{M}'_p \not\varepsilon \langle \mathbf{S}_i \rangle^R$, then $\sum_{p=i}^{n} \mathbf{M}'_p$ could not be equilibrated (i.e., eqn. (5.6) would be inconsistent). Taking the reciprocal product of both sides of eqn. (5.6) with $\mathbf{S}_i$, we obtain

$$\tau_j \mathbf{S}_j^a \circ \mathbf{S}_i + (\sum_{p=i}^{n} \mathbf{M}'_p) \circ \mathbf{S}_i = \mathbf{M}_i^c \circ \mathbf{S}_i \quad .$$

Since $\mathbf{S}_i^a \circ \mathbf{S}_i \neq 0$ and $\mathbf{M}_i^c \circ \mathbf{S}_i = 0$, we obtain

$$\tau = -\frac{(\sum_{p=i}^{n} \mathbf{M}'_p) \circ \mathbf{S}_i}{\mathbf{S}_i^a \circ \mathbf{S}_i} \quad . \tag{5.7}$$

We thus have a relation between the actuator wrench magnitudes $\tau_1, \tau_2, \ldots, \tau_n$ and the applied wrenches $\mathbf{M}'_1$, $\mathbf{M}'_2$, ..., $\mathbf{M}'_n$. If desired, the constraint wrenches $\mathbf{M}_1^c$, $\mathbf{M}_2^c$, ..., $\mathbf{M}_n^c$ can be subsequently determined from eqn. (5.6).

In industrial robots, it is very common that joints be either revolute or prismatic pairs. If $\mathbf{S}_i$ represents a revolute pair (thus, $\mathbf{S}_i$ is a rotor), then we shall always assume

$$S_i^a = S_i^\infty \quad ;$$

if S_i represents a prismatic pair (thus, S_i is improper), then we shall always assume S_i^a is a rotor such that

$$(S_i^a)^\infty = S_i \quad .$$

It is in this manner that industrial robots are usually actuated. We note that if S_i represents a revolute or prismatic pair, then $S_i^a \circ S_i = 1$, and eqn. (5.7) simplifies to

$$\tau_j = -(\sum_{i=j}^{n} M_i') \circ S_j \quad . \tag{5.8}$$

Equation (5.7) is the important result of this subsection. By using D'Alembert's principle, we can extend this equation so that a complete dynamic analysis can be performed for an open-loop chain. We consider this in the next chapter.

CHAPTER VI
DYNAMICS OF RIGID BODIES

In this chapter we tie together many of the results of previous chapters in order to derive the equations of motion for rigid bodies in terms of motors.

We first give a novel derivation of the inertia dyadic (the inertia dyadic was introduced by von Mises, 1924a). Moreover, by using a dyad expansion, we obtain a particularly simple form for this dyadic.

We then obtain the equations of motion for a rigid body; these equations are simply specializations of the equations for a particle system, which were given in Chapter IV.

Finally, the equations of motion for an actuated open-loop chain are derived. It is believed that these equations are simpler in form than any others found in the literature.

6.1 The Inertia Dyadic

6.1.1 Derivation

Let us consider a rigid body of mass m and let I^V be the inertia tensor about the mass center C. (The superscript in I^V indicates that it is a linear transformation for vectors.) In addition, let r_C be directed from an arbitrary point to the mass center. If the body has velocity motor $\mathbf{V}$ (with respect to an intertial frame), then the momentum motor is given by

$$\mathbf{H} : \mathbf{r}_C \times m\mathbf{v}_C + I^V\tilde{\mathbf{v}} \tag{6.1}$$

where $\tilde{h} = m\mathbf{v}_C$.

Let $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ be unit vectors each of which is in the direction of a principal axis (thus, these are the eigenvectors of I^V). With r_1, r_2, r_3 denoting the respective radii of gyration we have the following spectral decomposition:

$$I^V = m \sum_{i=1}^{3} r_i^2 \mathbf{p}_i \mathbf{p}_i \quad ,$$

where we assume that standard definition for vector dyads (i.e., $(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$). We substitute this dyadic and the identity dyadic $\sum_{i=1}^{3} \mathbf{p}_i \mathbf{p}_i$ into eqn. (6.1), which yields

$$\mathbf{H} : \mathbf{r}_C \times m\left(\sum_{i=1}^{3} \mathbf{p}_i \mathbf{p}_i\right)\mathbf{v}_C + m\left(\sum_{i=1}^{3} r_i^2 \mathbf{p}_i \mathbf{p}_i\right)\tilde{\mathbf{v}} \quad .$$

(The motivation for introducing these vector dyadics will be soon apparent.) Exploiting properties of vector dyads, we obtain

$$H : m \sum_{i=1}^{3} (\mathbf{r}_C \times \mathbf{p}_i)(\mathbf{p}_i \cdot \mathbf{v}_C) + m \sum_{i=1}^{3} r_i^2 \mathbf{p}_i (\mathbf{p}_i \cdot \tilde{\mathbf{v}}) \quad . \tag{6.2}$$

Now, we define $\mathbf{P}_i^0$ and $\mathbf{P}_i^\infty$ such that

$$\mathbf{P}_i^0 : \mathbf{r}_C \times \mathbf{p}_i \quad ,$$

$$\mathbf{P}_i^\infty : \mathbf{p}_i \quad ,$$

and we note that

$$\mathbf{p}_i \cdot \mathbf{v}_C = \mathbf{P}_i^0 \circ \mathbf{V} \quad ,$$

$$\mathbf{p}_i \cdot \tilde{\mathbf{v}} = \mathbf{P}_i^\infty \circ \mathbf{V} \quad .$$

Substituting into eqn. (6.2), we obtain

$$\mathbf{H} = m \sum_{i=1}^{3} \mathbf{P}_i^0(\mathbf{P}_i^0 \circ \mathbf{V}) + m \sum_{i=1}^{3} r_i^2\, \mathbf{P}_i^\infty(\mathbf{P}_i^\infty \circ \mathbf{V}) \quad . \tag{6.3}$$

Thus, $\mathbf{H}$ is a linear function of $\mathbf{V}$. Defining the motor dyadics,

$$\mathbf{I}^0 = m \sum_{i=1}^{3} \mathbf{P}_1^0\mathbf{P}_i^0 \quad , \tag{6.4}$$

$$\mathbf{I}^\infty = m \sum_{i=1}^{3} r_i^2\mathbf{P}_i^\infty\mathbf{P}_i^\infty \quad , \tag{6.5}$$

$$\mathbf{I} = m \sum_{i=1}^{3} (\mathbf{P}_i^0\mathbf{P}_i^0 + r_i^2\mathbf{P}_i^\infty\mathbf{P}_i^\infty) \tag{6.6}$$

we have that

$$\mathbf{H} = \mathbf{I}\mathbf{V} \tag{6.7}$$

where

$$\mathbf{I} = \mathbf{I}^0 + \mathbf{I}^\infty \quad .$$

We call $\mathbf{I}$ the <u>inertia dyadic</u> of the body. We note that

$$\mathbf{I}^0\mathbf{V} : \mathbf{r}_C \times m\tilde{\mathbf{v}}_C \quad , \tag{6.8}$$

$$\mathbf{I}^\infty\mathbf{V} : \mathbf{I}^V\tilde{\mathbf{v}} \quad . \tag{6.9}$$

It is clear that $\mathbf{I}^0$ is determined solely by the body's mass and the mass center and that $\mathbf{I}^\infty$ is determined solely by the mass distribution.

Finally, we note that $\{\mathbf{P}_1^0,\ \mathbf{P}_2^0,\ \mathbf{P}_3^0,\ \mathbf{P}_1^\infty,\ \mathbf{P}_2^\infty,\ \mathbf{P}_3^\infty\}$ is a standard basis fixed in the body frame. It is clear that the principal axes of inertia constitute the Cartesian coordinate system associated with this basis (i.e., the central axes of $\mathbf{P}_1^0$, $\mathbf{P}_2^0$, $\mathbf{P}_3^0$).

6.1.2 <u>Dyad Expansion</u>

A dyad expansion for an inertia dyadic $\mathbf{I}$ can be rapidly determined by using the following identity:

$$\sum_{i=1}^{3} (\mathbf{P}_i^0 \mathbf{P}_i^0 + r_i^2 \mathbf{P}_i^\infty \mathbf{P}_i^\infty)$$

$$= \frac{1}{2} \sum_{i=1}^{3} [(\mathbf{P}_i^0 + r_i \mathbf{P}_i^\infty)(\mathbf{P}_i^0 + r_i \mathbf{P}_i^\infty) +$$

$$(\mathbf{P}_i^0 - r_i \mathbf{P}_i^\infty)(\mathbf{P}_i^0 - r_i \mathbf{P}_i^\infty)] \quad . \tag{6.10}$$

Introducing the motors

$$\mathbf{P}_1 = \mathbf{P}_1^0 + r_1 \mathbf{P}_1^\infty \quad , \quad \mathbf{P}_2 = \mathbf{P}_1^0 - r_1 \mathbf{P}_1^\infty \quad ,$$

$$\mathbf{P}_3 = \mathbf{P}_2^0 + r_2 \mathbf{P}_2^\infty \quad , \quad \mathbf{P}_4 = \mathbf{P}_2^0 - r_2 \mathbf{P}_2^\infty \quad ,$$

$$\mathbf{P}_5 = \mathbf{P}_3^0 + r_3 \mathbf{P}_3^\infty \quad , \quad \mathbf{P}_6 = \mathbf{P}_3^0 - r_3 \mathbf{P}_3^\infty \quad ,$$

into identity (6.10) we can then rewrite definition (6.6) as

$$\mathbf{I} = \frac{m}{2} \sum_{i=1}^{6} \mathbf{P}_i \mathbf{P}_i \quad . \tag{6.11}$$

It is easily verified that the set $\{\mathbf{P}_i\}$ is co-reciprocal and that

$$\mathbf{P}_j \circ \mathbf{P}_j = 2h_j$$

where h_j is the pitch of $\mathbf{P}_j$ and

$$h_1 = r_1 \quad , \quad h_2 = -r_1 \quad ,$$

$$h_3 = r_2 \quad , \quad h_4 = -r_2 \quad ,$$

$$h_5 = r_3 \quad , \quad h_6 = -r_3 \quad .$$

In addition, $\{\mathbf{P}_i\}$ consists of eigenmotors, for,

$$\mathbf{IP}_j = \left(\frac{m}{2} \sum_{i=1}^{6} \mathbf{P}_i \mathbf{P}_i\right) \mathbf{P}_j$$

$$= \frac{m}{2} \mathbf{P}_j (\mathbf{P}_j \circ \mathbf{P}_j)$$

$$= m h_j \mathbf{P}_j \quad ;$$

the corresponding eigenvalues are

$$\lambda_1 = m r_1 \ (- \ m h_1) \quad , \quad \lambda_2 = -m r_1 \ (= m h_2) \quad ,$$

$$\lambda_3 = m r_2 \ (= m h_3) \quad , \quad \lambda_4 = -m r_2 \ (= m h_4) \quad ,$$

$$\lambda_5 = m r_3 \ (= m h_5) \quad , \quad \lambda_6 = -m r_3 \ (= m h_6) \quad .$$

We have tacitly assumed and we shall continue to assume that the radii of gyration (i.e., r_1, r_2, r_3) are all non-zero. It follows from this assumption that $\{\mathbf{P}_i\}$ is a basis of eigenmotors and that eqn. (6.11) gives the desired dyad expansion in which the simplification $\frac{\lambda_i}{\mathbf{P}_i \circ \mathbf{P}_i} = \frac{m}{2}$ has been incorporated.*

The screws $\langle \mathbf{P}_i \rangle$ are what Ball called the <u>principal screws of inertia</u> of a rigid body (Ball actually used the synthetic definition for the screw). It is clear that the motors $\mathbf{P}_i$ are fixed in the body.

Finally, using the dyad expansion for $\mathbf{I}$, we can immediately write down the dyad expansion for its inverse (see subsection 2.5.3):

$$\mathbf{I}^{-1} = \sum_{i=1}^{6} \frac{\lambda_i^{-1}}{\mathbf{P}_i \circ \mathbf{P}_i} \mathbf{P}_i \mathbf{P}_i$$

$$= \frac{1}{2m} \sum_{i=1}^{6} \frac{1}{h_i^2} \mathbf{P}_i \mathbf{P}_i \quad . \qquad (6.12)$$

---

* Equation (6.11) would still be correct even if some of the r_i were zero because identity (6.10) would still hold. It would not, however, be a dyad expansion according to our definition. For example, suppose $r_1 = 0$ and $r_2 \neq 0$, $r_3 \neq 0$. Then, $\mathbf{P}_1 = \mathbf{P}_2$ $(= \mathbf{P}_1^0)$ and eqn. (6.11) can be rewritten as

$$\mathbf{I} = m\mathbf{P}_1\mathbf{P}_1 + \frac{m}{2} \sum_{i=3}^{6} \mathbf{P}_i\mathbf{P}_i \quad .$$

The set $\{\mathbf{P}_i\}$ still comprises eigenmotors, but $\mathbf{I}$ cannot be expressed as a dyad expansion since $\mathbf{P}_1 \circ \mathbf{P}_1 = 0$.

6.1.3 The Intertia Inner Product

We now obtain a particularly useful inner product as will be demonstrated in section 6.3 where we obtain the equations of motion for an open-loop chain. This inner product is induced by an inertia dyadic, which we call the inertia inner product.

We show that an inertia dyadic **I** does, in fact, induce an inner product by using the dyad expansion:

$$\mathbf{I} = \frac{m}{2} \sum_{i=1}^{6} \mathbf{P}_i \mathbf{P}_i \quad .$$

Now, we must show that **I** is symmetric and that $\mathbf{M} \circ \mathbf{IM} > 0$ if and only if $\mathbf{M} \neq \mathbf{0}$. Since each dyad $\mathbf{P}_i \mathbf{P}_i$ is symmetric, it is clear that **I** is symmetric. In order to demonstrate the positive definiteness, let some motor **M** be expressed as a linear combination of the motors $\mathbf{P}_i$, that is, $\mathbf{M} = \sum_{i=1}^{6} \alpha_i \mathbf{P}_i$, and then we have

$$\mathbf{M} \circ \mathbf{IM} = \frac{m}{2} \sum_{i=1}^{6} (\mathbf{P}_i \circ \mathbf{M})^2$$

$$= \frac{m}{2} \sum_{i=1}^{6} (\mathbf{P}_i \circ \mathbf{P}_i)^2 \, \alpha_i^2$$

$$= \sum_{i=1}^{6} (m h_i^2) \, \alpha_i^2 \quad .$$

Since all $h_i \neq 0$, we have that $\sum_{i=1}^{6} (mh_i^2)\ \alpha_i^2 > 0$ if and only if not all $\alpha_i = 0$. Since not all $\alpha_i = 0$ is equivalent to $\mathbf{M} = 0$, we have that $\mathbf{M} \circ \mathbf{I}\mathbf{M} > 0$ if and only if $\mathbf{M} \neq 0$; thus, $\mathbf{I}$ induces an inner product $(\cdot|\cdot)$.

A useful expression for $(\mathbf{M}|\mathbf{N})$ can be obtained using eqns. (6.8) and (6.9). With C denoting the mass center of the body to which I corresponds, we have

$$\begin{aligned}\mathbf{M} \circ \mathbf{I}^0\mathbf{N} &= (\mathbf{M})_C \circ (\mathbf{I}^0\mathbf{N})_C \\ &= (\tilde{\mathbf{m}}, \mathbf{m}_C) \circ (m\tilde{\mathbf{m}}, \mathbf{0}) \\ &= m\mathbf{m}_C \circ \mathbf{n}_C\end{aligned}$$

and

$$\mathbf{M} \circ \mathbf{I}^\infty\mathbf{N} = \tilde{\mathbf{m}} \cdot \mathrm{I}^V\tilde{\mathbf{n}} \quad ;$$

thus,

$$\begin{aligned}(\mathbf{M}|\mathbf{N}) &= \mathbf{M} \circ (\mathbf{I}^0 + \mathbf{I}^\infty)\mathbf{N} \\ &= \mathbf{M} \circ \mathbf{I}^0\mathbf{N} + \mathbf{M} \circ \mathbf{I}^\infty\mathbf{N} \\ &= m\mathbf{m}_C \cdot \mathbf{n}_C + \tilde{\mathbf{m}} \cdot \mathrm{I}^V\tilde{\mathbf{n}} \quad .\end{aligned} \tag{6.13}$$

We call the norm induced by an inertia inner product and inertia norm.* We can give an interesting physical interpretation for the inertia norm. From Chapter IV, we have that the kinetic energy is given by

$$\underline{T} = \frac{1}{2}\,\mathbf{H} \circ \mathbf{V} \quad .$$

Now, since $\mathbf{H} = \mathbf{IV}$ we have

$$\begin{aligned} \underline{T} &= \frac{1}{2}\mathbf{V} \circ \mathbf{IV} \\ &= \frac{1}{2}\,\|\mathbf{V}\|^2 \quad . \end{aligned} \tag{6.14}$$

Finally, using, eqn. (6.13), we obtain the common expression

$$\underline{T} = \frac{1}{2}m\mathbf{v}_C \cdot \mathbf{v}_C + \frac{1}{2}\tilde{\mathbf{v}} \cdot \mathbf{I}^V\tilde{\mathbf{v}} \quad . \tag{6.15}$$

6.2 Equations of Motion for a Rigid Body

Before we derive the equations of motion for a rigid body, we need the following:

DEFINITION. A dyadic $\mathbf{A}$ is fixed in a frame if for every fixed motor $\mathbf{M}$, the motor $\mathbf{AM}$ is also fixed.

---

* In some advanced dynamic texts, the metric $d(\cdot,\cdot)$ such that $d(\mathbf{M},\mathbf{N}) = \|\mathbf{M}-\mathbf{N}\|$, where $\|\cdot\|$ is an inertia norm, is called the inertia metric.

It is clear that if the dyads that constitute a dyadic comprise fixed motors, then the dyadic is fixed. In particular, we have that the inertia dyadic of a body is fixed in the body frame.

Consider a rigid body for which **I** is the inertia dyadic, **V** is the velocity motor, and **M** the applied external wrench. Then substituting **H** = **IV** into the law of momentum (eqn. (4.1)), we obtain

$$\mathbf{M} = \frac{d}{dt}\mathbf{IV}$$

$$= \frac{d'}{dt}\mathbf{IV} + \mathbf{V} \times \mathbf{IV} \tag{6.16}$$

where $\frac{d'}{dt}$ denotes differentiation with respect to the body frame. Since **I** is fixed in the body frame, we have

$$\frac{d'}{dt}\mathbf{IV} = \mathbf{I}\frac{d'}{dt}\mathbf{V} \quad .$$

But

$$\frac{d'}{dt}\mathbf{V} = \dot{\mathbf{V}} - \mathbf{V} \times \mathbf{V}$$

$$= \dot{\mathbf{V}}$$

so that $\frac{d'}{dt}\mathbf{IV} = \mathbf{I}\dot{\mathbf{V}}$, and we can write the law of momentum for a rigid body as

$$\mathbf{M} = \mathbf{I}\dot{\mathbf{V}} + \mathbf{V} \times \mathbf{IV} \quad . \tag{6.17}$$

The law of conservation of momentum for rigid bodies is given by

$$(\mathbf{IV})\Big|_{t_o}^{t} = 0 \tag{6.18}$$

where $\mathbf{M} = \mathbf{0}$ over the interval $[t_o, t]$, and the principle of work and energy for rigid bodies is given by

$$\int_{t_o}^{t} \mathbf{M} \circ \mathbf{V} d\tau = \left(\tfrac{1}{2}\|\mathbf{V}\|^2\right)\Big|_{t_o}^{t} \quad . \tag{6.19}$$

We note that by using motor calculus, the equations of motion for a rigid body attain a very simple form. The above dynamical laws are essentially in the same form as given by von Mises (1924a and 1924b) (although, von Mises did not employ an inertia norm); our derivations, however, are quite different.

6.3 Dynamics of an Open-Loop Chain

The equations of motion for an open-loop chain are readily derived using D'Alembert's Principle and eqn. (5.7). We use the same notation as in Chapter V, and, in addition, we denote the momentum motor, inertia tensor, and energy inner product for the i^{th} link by $\mathbf{H}_i$, $\mathbf{I}_i$, and $(\cdot|\cdot)_i$. The intertial wrench "applied" to the i^{th} link is given by $-\dot{\mathbf{H}}_i$ where $\mathbf{H}_i = \mathbf{I}_i\mathbf{V}_i$. We obtain the requisite actuator torque/force magnitudes by extending eqn. (5.7) (i.e., by using D'Alembert's Principle):

$$\tau_i = -k_i\left[\sum_{p=i}^{n} (\mathbf{M}_p' - \dot{\mathbf{H}}_p)\right] \circ \mathbf{S}_i \tag{6.20}$$

where $k_i \equiv (\mathbf{S}_i^a \circ \mathbf{S}_i)^{-1}$. The constraint wrenches are then computed from from

$$\mathbf{M}_i^c = \dot{\mathbf{H}}_i - (\mathbf{M}_i' + \tau_i \mathbf{S}_i) \quad . \tag{6.21}$$

Since

$$\tau_i = -k_i\left(\sum_{p=i}^{n} \mathbf{M}_p'\right) \circ \mathbf{S}_i + k_i\left(\sum_{p=i}^{n} \dot{\mathbf{H}}_p\right) \circ \mathbf{S}_i \quad ,$$

it is clear that we can compute components of actuator torque/force magnitudes due to the $\mathbf{M}_p'$ and due to the $-\dot{\mathbf{H}}_p$ separately. We thus restrict ourselves to the dynamic problem; that is, we assume $\mathbf{M}_p' = \mathbf{0}$ for i = 1, 2, ..., n. In addition, we define

$$\tau_i' = \frac{\tau_i}{k_i} \tag{6.22}$$

(note that $\tau_i' = \tau_i$ for open-loop chains whose joints comprise only revolute and prismatic pairs). Then, eqn. (6.20) reduces to

$$\tau_i' = \left(\sum_{p=i}^{n} \dot{\mathbf{H}}_p\right) \circ \mathbf{S}_i \quad . \tag{6.23}$$

We now "expand" the right hand side of eqn. (6.23) in order to obtain the so-called explicit form, which is a form explicit in the $\dot{q}_i$ and $\ddot{q}_i$. Substituting for $\dot{\mathbf{H}}_p$ in eqn. (6.23) and distributing the reciprocal product, we obtain,

$$\tau_i' = \left[\sum_{p=i}^{n} (\mathbf{I}_p\dot{\mathbf{V}}_p + \mathbf{V}_p \times \mathbf{I}_p\mathbf{V}_p)\right] \circ \mathbf{S}_i$$

$$= \sum_{p=i}^{n} (\mathbf{S}_i \circ \mathbf{I}_p\dot{\mathbf{V}}_p + \mathbf{S}_i \circ \mathbf{V}_p \times \mathbf{I}_p\mathbf{V}_p) \quad .$$

Employing the identity $\mathbf{S}_i \times \mathbf{V}_p \circ \mathbf{I}_p\mathbf{V}_p = \mathbf{S}_i \circ \mathbf{V}_p \times \mathbf{I}_p\mathbf{V}_p$ and introducing the inertia inner product $(\cdot|\cdot)_p$ yields

$$\tau_i' = \sum_{p=i}^{n} [(\mathbf{S}_i|\dot{\mathbf{V}}_p)_p + (\mathbf{S}_i \times \mathbf{V}_p|\mathbf{V}_p)_p] \quad .$$

Substituting for $\mathbf{V}_p$ and $\dot{\mathbf{V}}_p$ using eqns. (5.2) and (5.4) and rearranging, we obtain

$$\tau_i' = \sum_{p=i}^{n} \left[\left(\mathbf{S}_i \middle| \sum_{j=1}^{p} \mathbf{S}_j\ddot{q}_j + \sum_{j=1}^{p} \sum_{k=1}^{j} \mathbf{S}_k \times \mathbf{S}_j\dot{q}_j\dot{q}_k\right)_p\right.$$

$$\left. + \left(\mathbf{S}_i \times \sum_{j=1}^{p} \mathbf{S}_j\dot{q}_j \middle| \sum_{k=1}^{p} \mathbf{S}_k\dot{q}_k\right)_p\right]$$

$$= \sum_{p=i}^{n} [\sum_{j=1}^{p} (\mathbf{S}_i | \mathbf{S}_j)_p \ddot{q}_j + \sum_{j=1}^{p} \sum_{k=1}^{j} (\mathbf{S}_i | \mathbf{S}_k \times \mathbf{S}_j) \dot{q}_j \dot{q}k$$

$$+ \sum_{j=1}^{p} \sum_{k=1}^{p} (\mathbf{S}_i \times \mathbf{S}_j | \mathbf{S}_k)_p \dot{q}_j \dot{q}_k] \quad . \tag{6.24}$$

The equations of motion can thus be expressed in the form

$$\tau_i^! = \sum_{j=1}^{n} D_{ij} \ddot{q}_j + \sum_{j=1}^{n} \sum_{k=1}^{n} C_{ijk} \dot{q}_j \dot{q}_k \quad . \tag{6.25}$$

The coefficients D_{ij} and C_{ijk}, determined from eqn. (6.24), can be expressed in the form

$$D_{ij} = \sum_{p=\max(i,j)}^{n} (\mathbf{S}_i | \mathbf{S}_j)_p \quad , \tag{6.26}$$

and

$$C_{ijk} = \sum_{p=\max(i,j,k)}^{n} [(\mathbf{S}_i \times \mathbf{S}_j | \mathbf{S}_K)_p - t_{jk} (\mathbf{S}_i | \mathbf{S}_j \times \mathbf{S}_k)_p] \tag{6.27}$$

where

$$t_{jk} = \begin{cases} 1 \text{ if } j > k, \\ 0 \text{ if } j < k \end{cases} \quad .$$

We can obtain an even simpler form for these coefficients by introducing an inner product $(\cdot | \cdot)_q^\star$ defined as follows:

$$(\mathbf{M}|\mathbf{N})^*_q = \sum_{p=q}^{n} (\mathbf{M}|\mathbf{N})_p \quad . \tag{6.28}$$

This new inner product is actually an inertia inner product induced by the composite inertia dyadic $\sum_{p=q}^{n} \mathbf{I}_p$. Substituting definition (6.28) into the expressions (6.26) and (6.27), we obtain

$$D_{ij} = (\mathbf{S}_i|\mathbf{S}_j)^*_p \quad , \quad p=\max(i,j) \tag{6.29}$$

and

$$C_{ijk} = (\mathbf{S}_i \times \mathbf{S}_j|\mathbf{S}_k)^*_p - t_{jk}(\mathbf{S}_i|\mathbf{S}_j \times \mathbf{S}_k)^*_p \quad , \tag{6.30}$$

$$p=\max(i,j,k) \quad .$$

Featherstone (1984, pp. 80-81) obtained essentially the same form for D_{ij} given by eqn. (6.29) in his "Composite-Rigid-Body Method." It is believed, however, that the form for C_{ijk} given by eqn. (6.30) is new.

Finally, we note that since C_{ijk} is the j,k^{th} entry of the matrix of a quadratic form, we can replace it with C'_{ijk} where for $j > k$,

$$C'_{ijk} = \frac{1}{2}(C_{ijk} + C_{ikj})$$

$$= \frac{1}{2}[(\mathbf{S}_i \times \mathbf{S}_j|\mathbf{S}_k)^*_p + (\mathbf{S}_k \times \mathbf{S}_j|\mathbf{S}_i)^*_p$$

$$(\mathbf{S}_i \times \mathbf{S}_k|\mathbf{S}_j)^*_p] \quad , \quad p = \max(i,j,k) \quad ,$$

and for $j < k$,

$$C'_{ijk} = C'_{ikj} \quad .$$

6.4 Discussion

The notion of coordinate independence is of central importance in this chapter. We did not introduce a coordinate system (of motors) in order to construct the inertia dyadic, nor did we introduce any coordinate systems in order to derive the equations of motion for an open-loop chain. It is believed that these coordinate independent derivations are new.

Algorithms for dynamics computations can be developed once coordinate systems are introduced. Ball (1900) expressed dynamic equations of a rigid body in terms of coordinates referred to a basis of eigenmotors of the body's inertia dyadic (the $\mathbf{P}_i$ in definition (6.1)). Most other researchers who have applied motor calculus or screw theory to rigid body dynamics have used standard bases (e.g., von Mises, 1924a and 1924b; Dimentberg, 1965; Yang, 1971; Woo and Freudenstein, 1971; and Featherstone, 1983 and 1984).

In order to develop efficient dynamics algorithms for open-loop chains, great care must be taken in the selection of both coordinate systems and the manner in which coordinates are transformed from one system to another. Featherstone (1984) has investigated this subject extensively.

There has been a vast amount of literature recently as regards the development of efficient dynamics algorithms for open-loop chains using vector methods (see, for example, Orin, McGhee, Vukobratovic, and Hartoch, 1979; Luh, Walker, and Paul, 1980; Walker and Orin, 1982; Thomas and Tesar, 1982; Renaud, 1984; Wang and Kohli, 1985; and Lathrop, 1985). Only Featherstone (1984) has considered the use of motor calculus in developing efficient algorithms.

A comparison of the forms of the equations of motion for an open-loop chain given in this chapter and in (Featherstone, 1984) with the equations given in the above cited references that use vector methods would illustrate the greatly simplified structure that results by using motor calculus. Perhaps an important contribution of this chapter is the simple forms obtained for the C_{ijk} coefficients.* The simplicity can be realized by comparing with the forms obtained in (Thomas and Tesar, 1982) and (Wang and Kohli, 1985). In both these papers, elaborate tabular schemes are necessary in order to determine these coefficients. The

---

* Featherstone (1984) did not determine expressions for these coefficients, as the algorithms that he developed did not require dynamic equations explicit in the joint velocities.

forms given in this chapter require no such schemes; the tables are, in effect, incorporated in the algebra.

CHAPTER VII
CONCLUSIONS

Motor calculus is clearly well-suited for rigid body dynamics. This has been demonstrated by von Mises (1924a and 1924b) and by several recent contributions (perhaps, most notably, Dimentberg, 1965 and Featherstone, 1983 and 1984). A main objective of this dissertation has been to treat motor calculus and its application to rigid body dynamics in a more general and systematic manner.

Important contributions include:

- the new motor definition,
- the development of a dyad expansion for a motor dyadic,
- a rigorous treatment of motor derivatives and integrals,
- a simple derivation of the circle representation for a two-system,
- simple derivations and representations of the inertia dyadic of a rigid body and the equations of motion for an open-loop chain.

It is hoped that some of the foundational areas presented in this dissertation will facilitate the many potential applications of motor calculus. Two suggestions for future work are the investigation of the dynamics of general mechanisms and further research on developing efficient algorithms for dynamic analyses.

REFERENCES

Ball, R. S., 1900, A Treatise on the Theory of Screws, Cambridge University Press, London.

Bottema, O., and Roth, B., 1979, Theoretical Kinematics, North Holland, Amsterdam, Holland.

Brand, L., 1947, Vector and Tensor Analysis, Wiley, New York.

Brand, L., 1957, Vector Analysis, Wiley, New York.

Clifford, W. K., 1873, "Preliminary Sketch of Bi-Quaternions," Proceedings of London Mathematical Society, Vol. 4, pp. 361-395.

Clifford, W. K., 1878, Elements of Dynamic, Macmillan, London.

Denavit, J., and Hartenberg, R. S., 1955, "A Kinematic Notation for Lower Pair Mechanisms Based on Matrices," Journal of Applied Mechanics, Vol. 22, pp. 215-221.

Dimentberg, F. M., 1965, The Screw Calculus and its Application in Mechanics (in Russian), Izdat. Nauka, Moscow, USSR. English Translation: AD 680993, Clearinghouse for Federal and Scientific Technical Information, 1968.

Duffy, J., 1980, Analysis of Mechanisms and Robot Manipulators, John Wiley and Sons, New York.

Everett, J. D., 1875, "On à New Method in Statics and Kinematics," Messenger of Mathematics, Vol. 4, pp. 36-83.

Featherstone, R., 1983, "The Calculation of Robot Dynamics using Articulated-Body Inertias," International Journal of Robotics Research, Vol. 2, No. 1, pp. 13-30.

Featherstone, R., 1984, "Robot Dynamics Algorithms" (Ph.D. Thesis), University of Edinburgh, Scotland.

Halmos, P. R., 1958, Finite-Dimensional Vector Spaces, Van Nostrand Reinhold Co., New York.

Hamilton, W. R., 1830, "First Supplement to an Essay on the Theory of Systems of Rays," Transactions of the Royal Irish Academy, Vol. 16, pp. 4-62.

Hiller, M., and Woernle, C., 1984, "A Unified Representation of Spatial Displacements," Mechanism and Machine Theory, Vol. 19, No. 6, pp. 477-486.

Hsia, L. M., and Yang, A. T., 1981, "On the Principle of Transference in Three-Dimensional Kinematics," Journal of Mechanical Design, Vol. 103, No. 3, pp. 652-656.

Hunt, K. H., 1978, Kinematic Geometry of Mechanisms, Oxford University Press, Oxford, Great Britain.

Huston, R. L., Passerello, C. E., and Harlow, M. W., 1978, "Dynamics of Multirigid-Body Systems," Journal of Applied Mechanics, Vol. 45, pp. 889-894.

Keler, M. L., 1973, "Kinematics and Statics Including Friction in Single-Loop Mechanisms by Screw Calculus and Dual Vectors," Journal of Engineering for Industry, Vol. 95, No. 2, pp. 471-480.

Keler, M. 1979, "Dual-Vector Half-Tangents for the Representation of Finite Motions of Rigid Bodies," Environment and Planning B, Vol. 6, pp. 403-412.

Kotelnikov, A. P., 1895, Vintovoye Schisleniye i Nekotoryye Prilozheniya Yego k Geometrii i Mekhanike (Screw Calculus and Some of its Applications to Geometry and Mechanics), Scientific Memoirs of the University of Kazan, Kazan.

Lang, S., 1966, Linear Algebra, Addison-Wesley, New York.

Lathrop, H. L., 1985, "Parallelism in Manipulator Dynamics," International Journal of Robotics Research, Vol. 4, No. 2, pp. 80-102.

Lewis, T. C., 1880, "On the Cylindroid," Messenger of Mathematics, Vol. 9, pp. 1-5.

Lipkin, H., and Duffy, J., 1982, "Analysis of Industrial Robots via the Theory of Screws," Proceedings of the 12th International Symposium on Industrial Robots, Paris, pp. 359-370.

Luh, J. Y. S., Walker, M. W., and Paul, R. P., 1980, "On-Line Computational Scheme for Mechanical Manipulators," Journal of Dynamic Systems, Measurement, and Control, Vol. 102, pp. 69-76.

Mises, R. v., 1924a, "Motorrechnung, ein neues Hilfsmittel der Mechanik" (Motor Calculus: A New Tool for Mechanics), Zeitschrift fur Angewandte Mathematik und Mechanik, Vol. 4, No. 2, pp. 155-181.

Mises, R. v., 1924b, "Anwendungen der Motorrechnung" (Applications of Motor Calculus), Zeitschrift fur Angewandte Mathematic und Mechanik, Vol. 4, No. 3, pp. 193-213.

Orin, D. E., McGhee, R. B., Vukobratovic, M., and Hartoch, G., 1979, "Kinematic and Kinetic Analysis of Open-Chain Linkages Utilizing Newton-Euler Methods," Mathematical Biosciences, Vol. 43, pp. 107-130.

Pennock, G. R. and Yang, A. T., 1983, "Dynamic Analysis of a Multi-Rigid-Body Open-Chain System," Journal of Mechanisms, Tranmissions, and Automation in Design, Vol. 105, No. 1, pp. 28-34.

Phillips, J., 1984, Freedom in Machinery, Cambridge University Press, Cambridge, Mass.

Phillips, J. R., and Hunt, K. H., 1964, "On the Theorem of Three Axes in the Spatial Motion of Three Bodies," Australian Journal of Applied Science, Vol. 15, No. 4, pp. 267-287.

Plücker, J., 1866, "Fundamental Views Regarding Mechanics," Philosophical Transactions, Vol. 156, pp. 361-380.

Renaud, M., 1984, "An Iterative Analytical Procedure for Obtaining a Robot Manipulator Dynamic Model," in Brady, M., and Paul, R., ed., Robotics Research: The First International Symposium, MIT Press, Cambridge, Mass., pp. 749-764.

Rooney, J., 1975, "On the Principle of Transference," Proceedings of the Fourth World Congress on the Theory of Machines and Mechanisms, Newcastle upon Tyne, Vol. 5, pp. 1089-1094.

Roth, B., 1984, "Screws, Motors, and Wrenches That Cannot Be Bought in a Hardware Store," in Brady, M., and Paul, R., ed., Robotic Research: The First International Symposium, MIT Press, Cambridge, Mass., pp. 679-693.

Study, E., Geometrie der Dynamen (Geometry of Dynamics), 1903, B. G. Teubner, Leipzig.

Sugimoto, K., and Duffy, J., 1982, "Application of Linear Algebra to Screw Systems," Mechanism and Machine Theory, Vol. 17, No. 1, pp. 73-83.

Thomas, M., and Tesar, D., 1982, "Dynamic Modeling of Serial Manipulator Arms," Journal of Dynamic Systems, Measurements and Control, Vol. 104, No. 3, pp. 218-228.

Veldkamp, G. R., 1976, "On the Use of Dual Numbers, Vectors, and Matrices in Instantaneous Spatial Kinematics," Mechanism and Machine Theory, Vol. 11, No. 2, pp. 141-156.

Voyevodin, V. V., 1983, Linear Algebra, Shokurov, V., translator, Mir, Moscow.

Waldron, K. J., 1972, "A Method of Studying Joint Geometry," Mechanism and Machine Theory, Vol. 7, pp. 347-353.

Walker, M. W. and Orin, D. E., 1982, "Efficient Dynamic Computer Simulation of Robot Mechanisms," Journal of Dynamic Systems, Measurement, and Control, Vol. 104, pp. 205-211.

Wang, T., and Kohli, D., 1985, "Closed and Expanded Form of Manipulator Dynamics Using Lagrangian Approach," Journal of Mechanisms, Transmissions, and Automation in Design, Vol. 107, No. 2, pp. 223-225.

Woo, L., and Freudenstein, F., 1970, "Application of Line Geometry to Theoretical Kinematics and the Kinematic Analysis of Mechanical Systems," Journal of Mechanisms, Vol. 5, pp. 417-460.

Woo, L., and Freudenstein, F., 1971, "Dynamic Analysis of Mechanisms Using Screw Coordinates," Journal of Engineering for Industry, Vol. 93, pp. 273-276.

Yang, A. T., 1967, "Application of Dual Quaternions to the Study of Gyrodynamics," Journal of Engineering for Industry, Vol. 89, No. 1, pp. 137-143.

Yang, A. T., 1969, "Analysis of an Offset Unsymmetric Gyroscope with Oblique Rotor Using (3x3) Matrices with Dual-Number Elements," Journal of Engineering for Industry, Vol. 91, No. 3, pp. 535-542.

Yang, A. T., 1971, "Inertia Force Analysis of Spatial Mechanisms," Journal of Engineering for Industry, Vol. 93, pp. 27-32.

Yang, A. T., 1974, "Calculus of Screws," Basic Questions of Design Theory, North Holland, Amsterdam, pp. 266-281.

Yang, A. T., and Freudenstein, F., 1964, "Application of Dual-Number Quaternion Algebra to the Analysis of Spatial Mechanisms," Journal of Applied Mechanics, Vol. 86, pp. 300-308.

BIOGRAPHICAL SKETCH

Gilbert H. Lovell III was born on February 7, 1957, in Syracuse, New York. He received the Bachelor of Science in Mechanical Engineering degree from Rensselaer Polytechnic Institute in May, 1979, and he began graduate studies in mechanical engineering at the University of Florida in September, 1979, funded by a research assistantship.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Joseph Duffy

Dr. Joseph Duffy, Chairman
Professor of Mechanical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Gary K Matthew

Dr. Gary K. Matthew
Associate Professor of Mechanical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Dr. Ralph G. Selfridge
Professor of Computer and Information Sciences

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

John Staudhammer

Dr. John Staudhammer
Professor of Electrical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Chang-Tsan Sun

Dr. Chang-Tsan Sun
Professor of Engineering Sciences

This dissertation was submitted to the Graduate Faculty of the College of Engineering and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

May 1986

W G Shafer
Dean, College of Engineering

Dean, Graduate School