On the Complexity of Smooth Spline Surfaces from Quad Meshes

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Abstract

A standard task of CAGD is to determine G^1-connected tensor-product B-spline patches approximating a quadrilateral mesh whose vertices can have any fixed valence. This paper derives fundamental relations that sufficiently smooth regular patches have to obey to be part of a C^1 complex and that restrict the choice of G^1-reparameterizations when patches are computed by local averaging.

In particular, when one patch is associated with each quadrilateral facet, and the patch is a polynomial spline of degree bi-3, then the relations dictate the minimal number and multiplicity of knots: for general data and geometric flexibility, there must be at least two internal knots per edge and at least one must be a double knot.

1 Introduction

Quad(rilateral) meshes are used in geometric design and computer graphics, because they capture symmetries of natural and man-made objects. Smooth surfaces of degree bi-3 can be generated by applying subdivision to the quad mesh [CC78] or, alternatively, by joining a finite number of polynomial pieces (see e.g. [Pet00]). When quads form a checkerboard arrangement, we can interpret 4×4 grids of vertices as control points of bi-3 B-spline surface. Then we call the central quad *ordinary* and are guaranteed that adjacent ordinary quad patches join C^2.

The essential challenge comes from covering *extraordinary* quads, i.e. quads that have one or more vertices of valence $n \neq 4$. While this can be addressed by recursive subdivision, representing the surface with a finite small number of patches is often preferable. In particular, to take advantage of parallelism in the construction, such as GPU acceleration (see e.g. [LS08, NYM$^+$08]), finite, localized, parallelizable construction steps are needed. That is, the construction near vertices must dependent only on a fixed, small neighborhood of the vertex and the remaining construction for each quad depends only on the newly computed vertex data in a small neighborhood of the quad.

This raises the fundamental question, Q, answered in this paper: *what is the simplest structure* (least number of knots) *of degree bi-3 spline patches that allow a general quad mesh, including extraordinary quads, to be converted by localized operations into a smooth surface with one spline patch per quad?*

Section 2 takes a more general view. Here we do not constrain the domain to be a collection of quadrilaterals or the functions to be polynomial splines. The relations of Lemma 1, 2 and 3 in main part also do not depend on locality of the construction but apply to any collection of sufficiently smooth patches coming together with an unbiased, logically symmetric G^1 join (Definition 1, 3). Adding the locality requirement in Section 2.1 allows us to rule out certain G^1 reparameterizations.

In Section 3, we specialize the setting to polynomial tensor-product splines of degree bi-3. For these, we deduce a lower bound on the number and multiplicity of knots. We prove that at least two internal knots are required per edge and at least one must be a double knot to admit a local construction. These lower bounds are tight since constructions for smooth surfaces (without the shape defects characterized in Lemma 5) exist for the minimal cases of exactly two internal knots [FP08] (and for one internal double knot and several single knots). Together, these lower and upper bounds (algorithms) conclusively settle the question Q.

1.1 Bi-3 constructions in the literature

Creating C^1 surfaces of degree bi-3, i.e. generalizing standard tensor-product B-spline layout to arbitrary manifold quad meshes with first-order differentiability, is a classic challenge of CAGD (see e.g. [Bez77, vW86, Pet91]). More recently, a number of papers appeared that are predicated on the conclusion that simple constructions are not possible. PCCM [Pet00] generates smooth bi-3 surfaces but requires up to two steps of Catmull-Clark subdivision to separate non-4-valency vertices. This proves that a 4×4 arrangement of polynomial patches per quad, corresponding to two internal double knots and one single knot, suffices in principle. However, [Pet01a] pointed out that PCCM can foce poor shape for certain higher-order saddles (see also Lemma 5). Similarly, [SWWL04] justifies a subdivision-like refinement approach with C^2 bi-3 tensor-product patches to obtain approximatly smooth surfaces by, correctly as we will see, stipulating that no finite exact construction is possible. Loop and Schaefer[LS08] propose a bi-3 C^0 surface construction with separate tangent patches to convey the impression of smoothness when applying lighting, a technique cwproposed for computer graphics in [VPBM01]. Again, the results in this paper justifies the approach if one restricts each patch to be a cubic polynomial. For the same reason, Myles *et al.* [MYP08] perturb a bi-3 base patch near non-4-valency vertices to obtain a smooth surface suitable for CAD applications with one bi-5 patch per quad. As mentioned, [FP08] gives an algorithm that constructs smoothly connected Bézier patches of degree bi-3 and such that the internal transitions allow re-interpretion as tensor-product spline patches with two internal double knots. Related, but structurally different, in the polar patch layout collapsed bi-3 patches allow for a simple C^1 surface construction also for high valences [MKP07].

2 G^1 continuity

Consider n parameterically C^1 patches

$$\mathbf{b}^k : \square \subsetneq \mathbb{R}^2 \to \mathbb{R}^3, \qquad k = 1, \ldots, n \tag{1}$$

meeting at a central point $\mathbf{b}^k(0,0) = \mathbf{p}$ such that $\mathbf{b}^k(u,0) = \mathbf{b}^{k-1}(0,u)$. We do not (yet) assume that $\square$ is the unit square but just that the origin as a corner with two independent edge directions emanating from it. We assume the patches are not singular at the origin in the sense that $\partial_2 \mathbf{b}^k(0,0) \times \partial_1 \mathbf{b}^k(0,0) \neq 0$ where ∂_ℓ denotes differentiation with respect to the ℓth argument.

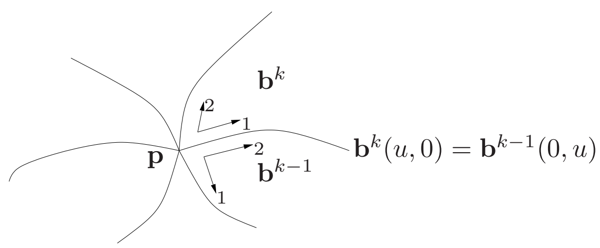

Figure 1: **Indexing and parameterization** of adjacent patches at a vertex of valence n. Here $\mathbf{b}^{k-1} = \mathbf{b}^n$ if $k = 1$.

To make the n patches form a C^1 surface, we want to enforce logically symmetric (unbiased) G^1 constraints.

Definition 1 (Unbiased G^1 constraints) *With α^k a sufficiently smooth, univariate* scalar-valued function, *the* unbiased G^1 *constraints between consecutive patches are*

$$\partial_2 \mathbf{b}^k(u,0) + \partial_1 \mathbf{b}^{k-1}(0,u) = \alpha^k(u) \partial_1 \mathbf{b}^k(u,0), \tag{2}$$

If $\alpha^k \equiv 0$, the constraints enforce parametric C^1 continuity. *We abbreviate the ℓth derivative of α^k evaluated at 0 as a^k_ℓ.*

Justification We can derive lower bound statements from this special unbiased setting since an algorithm must work in general, i.e. for all data, and therefore also for input whose symmetries suggest an unbiased treatment, e.g. no dependence on the indexing or ordering of patches and no bias based on the geometric interpretation.

We now add the assumption that each $\mathbf{b}^k$ is twice continuously differentiable at $(0,0)$ (as are the polynomial pieces of a spline patch).

Definition 2 (Spline) *A C^s spline patch is a patch* $\mathbf{b}$ *that is internally parameterically C^s, in particular across finitely many (domain knot) lines. At any boundary knot (intersection of a knot line with the boundary of the domain), on either side of the knot line, $\partial_1^i \partial_2^j \mathbf{b}^k$) is well-defined for $i + j \le s + 1$. And the patch is regular in the corners: $\partial_2 \mathbf{b}^k(0, 0) \times \partial_1 \mathbf{b}^k(0, 0) \ne 0$.*

We can then differentiate along (the respective domain edge of) the common boundary $\mathbf{b}^k(u, 0) = \mathbf{b}^{k-1}(0, u)$:

$$(\partial_1 \partial_2 \mathbf{b}^k)(u, 0) + (\partial_2 \partial_1 \mathbf{b}^{k-1})(0, u) = \alpha^k(u) \partial_1^2 \mathbf{b}^k(u, 0) + (\alpha^k)'(u) \partial_1 \mathbf{b}^k(u, 0). \quad (3)$$

When we evaluate at $u = 0$ then

$$\text{at } (0, 0), \qquad \partial_1 \partial_2 \mathbf{b}^k + \partial_2 \partial_1 \mathbf{b}^{k-1} = a_0^k \partial_1^2 \mathbf{b}^k + a_1^k \partial_1 \mathbf{b}^k. \quad (4)$$

If n is *even* then the alternating sum of the left hand sides vanishes,

$$\text{at } (0, 0), \qquad \sum_{k=1}^{n} (-1)^k \big(\partial_1 \partial_2 \mathbf{b}^k + \partial_2 \partial_1 \mathbf{b}^{k-1}\big) = 0 \quad (5)$$

and therefore so must the right hand side

$$\text{at } (0, 0), \qquad 0 = \sum_{k=1}^{n} (-1)^k a_0^k \partial_1^2 \mathbf{b}^k + \sum_{k=1}^{n} (-1)^k a_1^k \partial_1 \mathbf{b}^k. \quad (6)$$

In particular, if the patches join smoothly and therefore have a unique normal $\mathbf{n} \in \mathbb{R}^3$ at $\mathbf{p}$ then , with $\cdot$ denoting scalar product,

$$\text{if } n \text{ is even, at (0,0)} \qquad 0 = \sum_{k=1}^{n} (-1)^k a_0^k \ \mathbf{n} \cdot \partial_1^2 \mathbf{b}^k. \quad (7)$$

This is the *vertex-enclosure constraint* (see e.g. [Pet02, p.205]).

We now focus on the case $n = 4$.

Definition 3 (tangent X) *If $n = 4$, $\partial_1 \mathbf{b}^1(0, 0) = -\partial_1 \mathbf{b}^3(0, 0)$ and $\partial_1 \mathbf{b}^2(0, 0) = -\partial_1 \mathbf{b}^4(0, 0)$ then the tangents form an X.*

Lemma 1 (tangent X) *If the tangents form an X, then $a_1^1 = a_1^3$ and $a_1^2 = a_1^4$.*

Proof If the tangents form an X then $n = 4$ and $a_0^k = 0$, $k = 1, 2, 3, 4$ so that (6) simplifies to

$$\text{at (0,0)}, \qquad 0 = (a_1^1 - a_1^3) \partial_1 \mathbf{b}^1 - (a_1^2 - a_1^4) \partial_1 \mathbf{b}^2. \quad (8)$$

Since the patches are not degenerate, both summands have to vanish, implying the claim. |||

We now consider the unbiased G^1 transition between two C^1 spline patches. We focus on an *interior boundary vertex* (corresponding to an *interior knot* on the common boundary). That is, we consider a point where four polynomial pieces meet such that $\mathbf{b}^1$ and $\mathbf{b}^2$ belong to one spline patch and $\mathbf{b}^3$ and $\mathbf{b}^4$ are adjacent pieces of the edge-adjacent spline patch (Figure 2). Since each spline patch is internally parameterically C^1,

$$\alpha^2 \equiv 0 \equiv \alpha^4. \tag{9}$$

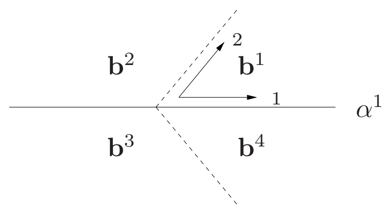

Figure 2: **Interior knot join** on the boundary (solid) between two splines. The first spline has polynomial pieces $\mathbf{b}^1$ and $\mathbf{b}^2$.

Lemma 2 (C^1 spline, interior knot) *Let* $(0,0)$ *be the parameter associated with an interior knot on the boundary common to two* C^1 *splines joined by unbiased* G^1 *constraints. Then*

$$a_0^1 = -a_0^3, \tag{10}$$

$$\textit{at } (0,0): \ 0 = a_0^1(\partial_1^2\mathbf{b}^1 - \partial_1^2\mathbf{b}^3) + (a_1^1 - a_1^3)\mathbf{t}_1, \quad \mathbf{t}_k := \partial_1\mathbf{b}^k(0,0). \tag{11}$$

Proof The parametric C^1 constraints imply $\partial_1\mathbf{b}^1(0,0) = -\partial_1\mathbf{b}^3(0,0)$ and since $n = 4$, (10) follows. By (9), (6) specializes to

$$\begin{aligned} \text{at } (0,0), \qquad 0 &= a_0^1\partial_1^2\mathbf{b}^1 + a_0^3\partial_1^2\mathbf{b}^3 + a_1^1\partial_1\mathbf{b}^1 + a_1^3\partial_1\mathbf{b}^3 \\ &= a_0^1(\partial_1^2\mathbf{b}^1 - \partial_1^2\mathbf{b}^3) + (a_1^1 - a_1^3)\mathbf{t}_1 \end{aligned} \tag{12}$$

as claimed. |||

So, remarkably, when two spline patches meet along a common boundary, unbiased G^1 constraints *across* this boundary imply the constraint (11) exclusively in terms of derivatives *along* the boundary.

Lemma 3 (C^2 spline, interior knot) *Let* $(0,0)$ *be the parameter associated with an interior knot on the boundary common to two* C^2 *splines joined by unbiased* G^1 *constraints. Then, in addition to* (10),

$$a_1^1 = a_1^3, \tag{13}$$

$$\textit{at } (0,0): \ 0 = a_0^1(\partial_1^3\mathbf{b}^1 - \partial_1^3\mathbf{b}^3) + 4a_1^1\partial_1^2\mathbf{b}^1 + (a_2^1 - a_2^3)\mathbf{t}_1. \tag{14}$$

Proof Since the splines are C^2, $\partial_1^2\mathbf{b}^1(0,0) = \partial_1^2\mathbf{b}^3(0,0)$. Then (6) implies (13).

By assumption, for the spline-internal boundary,

$$\text{for } k = 2, 4, \text{ at } (0,0), \qquad \partial_2\partial_1\partial_2\mathbf{b}^k + \partial_1\partial_2\partial_1\mathbf{b}^{k-1} = 0. \tag{15}$$

Differentiating (3) once more along the (direction corresponding to the) common boundary, we obtain for

$$k = 1, 3, \text{at } (0,0), \quad \partial_1\partial_1\partial_2\mathbf{b}^k + \partial_2\partial_2\partial_1\mathbf{b}^{k-1} = a_0^k\partial_1^3\mathbf{b}^k + 2a_1^k\partial_1^2\mathbf{b}^k + a_2^k\partial_1\mathbf{b}^k. \tag{16}$$

Summing the two instances of (16) and subtracting the two instances of (15) eliminates the mixed derivatives and yields at $(0,0)$

$$0 = a_0^1\partial_1^3\mathbf{b}^1 + 2a_1^1\partial_1^2\mathbf{b}^1 + a_2^1\partial_1\mathbf{b}^1 + a_0^3\partial_1^3\mathbf{b}^3 + 2a_1^3\partial_1^2\mathbf{b}^3 + a_2^3\partial_1\mathbf{b}^3. \tag{17}$$

Then C^2 continuity implies (14). |||

2.1 Vertex-localized construction and linear α

Equation (4) shows that the Taylor expansions up to order two of the patches joining at a point are strongly intermeshed in the sense that they constrain a common set of derivatives. So, to avoid having to solve a global system, we define the following.

Definition 4 (vertex-localized construction) *A construction is* G^1 vertex-localized *if we can solve the unbiased* G^1 *constraints* (2) *and* (3) *on the second-order Taylor expansion* $\partial_1^i\partial_2^j\mathbf{b}^k$, $0 \le i, j, i+j \le 2$ *at* $(0,0)$ *corresponding to a vertex, independent of the solutions at its neighbors.*

Note that we allow the local solutions to depend on the a priori known local connectivity and the valence of the neighbors in particular but not on the local patch expansion computed there. In the most challenging case, the vertex enclosure constraint (7) applies at each vertex. Therefore a vertex-localized construction implies that the *second-order Taylor expansion* is set independently at each vertex of the quad mesh. It is to this scenario and the *unrestricted choice of geometric instantiation* that we take into account, when we prefix a statement with **in general**.

Along a boundary, we index the ℓth derivative of the scalar map in the G^1 constraints of the jth spline piece meeting, as $a_{\ell,j}$.

Lemma 4 (piecewise linear α) *In general, a vertex-localized construction of unbiased* G^1 *transitions between* C^1 *spline patches with* everywhere *at most linear* α *is not possible.*

Proof Consider a vertex surrounded by vertices of valence $n = 4$. Then vertex-localized construction implies that $a_{0,0} = 0$. Assume now that immediate neighbor interior knots exist. Then local construction implies also $a_{0,-1} = 0$ (and $a_{0,1} = 0$) for these interior boundary vertexs. Shifting the focus to one such

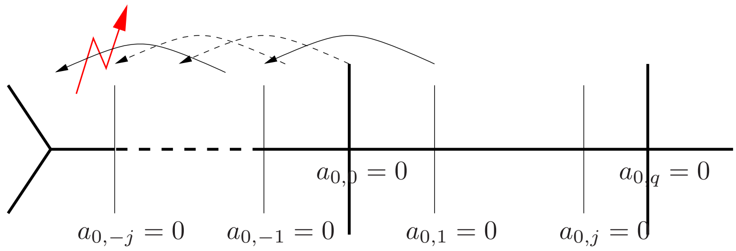

Figure 3: **Propagation** of $a_{0,j} = 0$ in Lemma 4.

interior boundary vertex, say the one corresponding to $a_{0,-1} = 0$, we observe that its tangents form an X since the two splines, one at either side, are C^1. So Lemma 1 implies $a_{0,-2} = 0$ since $a_{0,0} = 0$ and again this interior boundary vertex's tangents form an X. In this manner, X formations and $a_{0,-j} = 0$ propagate (see the arrows in Figure 3 for illustration). Once the propagation meets an original vertex with valence $n \neq 4$ (whether or not we have interior knots), local construction clashes with Lemma 1. $|||$

Unbiased G^1 constraints imply a *local, unbiased choice of the tangent directions*, namely such that (see [Pet94, Prop 3])

$$\alpha^k(0) := 2\cos\frac{2\pi}{n}. \tag{18}$$

Corollary 1 *If $n = 4$, a local, unbiased choice of the tangent directions and α^k linear imply $n^k = n^{k+2}$.*

Proof The claim follows from Lemma 1 since by the unbiased choice $\alpha^k(0) := 0$ and $\alpha^k(1) := 2\cos\frac{2\pi}{n^k}$. $|||$

Lastly, we characterize a source of poor shape in smooth surface constructions [Pet01b].

Lemma 5 (Flatness at saddle points) *A curve segment $\mathbf{c}$ of an unbiased construction, emanating from a higher-order saddle point $\mathbf{p} := \mathbf{c}(0)$ and such that*

$$\mathbf{c}' := \boldsymbol{\ell}\gamma, \quad \boldsymbol{\ell} : \mathbb{R} \to \mathbb{R}^3,\ deg(\boldsymbol{\ell}) \leq 1, \quad \gamma : \mathbb{R} \to \mathbb{R},\ deg(\gamma) \leq 1, \tag{19}$$

is planar. If the saddle is symmetric, $\mathbf{c}$ is a straight line segment.

Proof Let $\mathbf{n}$ be the normal at $\mathbf{p}$ and, without loss of generality, $\gamma(u) := 1 + \gamma_1 u$ for some $\gamma_1 \in \mathbb{R}$. Then $\mathbf{c}'(0) = \boldsymbol{\ell}(0)$, $\mathbf{c}''(0) = \boldsymbol{\ell}'(0) + \boldsymbol{\ell}(0)\gamma_1$ and $\mathbf{c}'''(0) = 2\boldsymbol{\ell}'(0)\gamma_1$. At a higher-order saddle point, the normal curvature is zero, and therefore $\mathbf{n}\mathbf{c}''(0) = 0$. This implies $\mathbf{n}\boldsymbol{\ell}'(0) = 0$ and $\mathbf{n}\mathbf{c}'''(0) = 0$. If the saddle is symmetric then $\mathbf{c}'(0)$ and $\mathbf{c}''(0)$ are collinear and so is $\mathbf{c}'''(0) = 2\boldsymbol{\ell}'(0)\gamma_1$. $|||$

A higher-order saddle points, such as the center of the monkey saddle, should have non-zero Gauss curvature apart from the point. Therefore, we will later *disqualify* constructions that force *straight boundary segments* for such non-flat geometry.

3 Lower bounds for degree bi-3

We now argue that, in general, vertex-localized construction with splines of degree bi-3 (bicubic) is possible only if the spline patches have at least two internal knots per edge and at least one that is a double knot.

Since we specialize to polynomial $\mathbf{b}^k$ of polynomial degree 3, a simple algebraic argument [Pet91, Cor. 2.2] shows that α is a rational function, $\alpha =: \frac{\beta}{\gamma}$, with numerator of polynomial degree $\deg(\beta) \le 6$ and $\deg(\gamma) \le \deg(\beta) - 1$ and such that β and γ have no factor in common. In fact, for unbiased constructions we can conclude a lower bound on the degree.

Lemma 6 (α restricted) *If the two patches $\mathbf{b}^k$ and $\mathbf{b}^{k-1}$ satisfying an unbiased G^1 constraint are of degree bi-3 and the boundary segment $\mathbf{b}^k(u,0)$ should not be forced to be a line segment then*

$$\alpha := \frac{\beta}{\gamma}, \quad (deg(\beta), deg(\gamma)) \in \{(2,1),(2,0),(1,1),(1,0),(0,0)\} \text{ and } \tag{20}$$

$$\partial_1 \mathbf{b}^k(u,0) = \boldsymbol{\ell}(u)\gamma(u), \quad deg(\boldsymbol{\ell}) = 2 - deg(\gamma). \tag{21}$$

Proof We may assume that β and γ are relatively coprime. Then the unbiased G^1 constraint (2) implies that γ is a (scalar) factor of $\partial_1 \mathbf{b}^k(u,0) \in \mathbb{R}^3$, the (vector-valued) derivative of the boundary. By the assumptions, $0 < \deg(\partial_1 \mathbf{b}^k(u,0)) \le 2$. Since $\deg(\partial_2 \mathbf{b}^k(u,0) + \partial_1 \mathbf{b}^{k-1}(0,u)) \le 3$ this limits $\deg(\beta) \le 2$ and $\deg(\gamma) \le 1$. |||

Scaling numerator and denominator, we may assume that $\gamma(u) := 1 + \gamma_1 u$.

Corollary 2 (polynomial vertex α) *For a boundary segment of degree 3 directly attached to a vertex, α is linear.*

Proof Choosing $(\deg(\beta), \deg(\gamma)) \in \{(2,1),(2,0),(1,1)\}$ forces the degree 3 curve segment to be of the form (19). Since we require in general more flexibility than forced straight line segments, Lemma 5 shows that α must be linear or constant for segments emmanating from the (original) vertices. |||

Now recall that vertex-localized construction in general implies that position, first and second derivative are specified at each endpoint of a (piecewise) G^1 join between splines.

3.1 No internal knot

Without internal knot each $\mathbf{b}^k$ is a Bézier patch and its boundary curve a single polynomial piece of degree 3. The second order expansion at a point $\mathbf{p}$ interferes with the expansion at its neighbor. In particular, for valence $n = 2m > 4$, Equation (7) links the second order Taylor expansions of all direct neighbors, preventing a vertex-local construction.

3.2 1-fold internal knots only

Since the knots are of multiplicity 1 and the degree is bi-3, we have two C^2 spline patches meeting along a common boundary. Then each additional interior knot corresponds to one additional boundary curve segment of degree 3. However, C^0, C^1 and C^2 constraints plus (14) of Lemma 3 impose four vector-valued constraints. This is the intuition underlying the next lemma.

Lemma 7 (1-fold knots only) *In general, no vertex-localized construction enforcing unbiased G^1 constraints is possible using only C^2 splines of degree bi-3.*

Proof For the first and the last segment of the piecewise cubic spline boundary curve, position, first and second derivative are given by vertex-local construction. It is easy to check that if the boundary curve has two segments, there are not enough degrees of freedom to, in general, enforce C^2 continuity, but for three segments C^2 continuity uniquely determines all segments. However, this construction leaves (14) unresolved at the two interior knots and therefore, in general, yields no solution to (2). Inserting an additional knot adds another curve segment. Let segment $\mathbf{b}^3(u,0)$ be given and $\mathbf{b}^1(u,0)$ the result of adding a knot (the segments are arranged as in Figure 2). By Lemma 4, not all α can be linear. (Anyhow, if α is linear, it is fixed by (10) and (13) and therefore the free (B-spline) control point must be used to resolve (14). That is, we do not gain degrees of freedom to enforce (2).)

We therefore assume that $(\deg(\beta), \deg(\gamma)) \in \{(2,1),(2,0),(1,1)\}$. Then $\partial_1 \mathbf{b}^1(u,0) := \boldsymbol{\ell}(u)\gamma(u)$, a linear vector-valued polynomial times the scalar (possibly constant) factor $\gamma(u) := 1+\gamma_1 u$. By (10) and (13) and the C^2 constraints, constraint (14) becomes at $(0,0)$

$$0 = a_0^3(\underbrace{\partial_1^3\mathbf{b}^3 - 2\gamma_1\boldsymbol{\ell}'(0)}_{=:\mathbf{v}}) + 4a_1^3\partial_1^2\mathbf{b}^3 + (a_2^3 - a_2^1)\mathbf{t}_3. \tag{22}$$

By C^1 continuity $\boldsymbol{\ell}(0)\gamma(0) = \boldsymbol{\ell}(0) = -\mathbf{t}_3$ and hence the C^2 constraint $\partial_1^2\mathbf{b}^3 = \boldsymbol{\ell}(0)\gamma_1 + \boldsymbol{\ell}'(0) = -\mathbf{t}_3\gamma_1 + \boldsymbol{\ell}'(0)$ implies

$$\boldsymbol{\ell}'(0) = \mathbf{t}_3\gamma_1 + \partial_1^2\mathbf{b}^3(0,0). \tag{23}$$

Therefore, at $(0,0)$, $\mathbf{v} = \partial_1^3\mathbf{b}^3 - 2\gamma_1(\mathbf{t}_3\gamma_1 + \partial_1^2\mathbf{b}^3)$. Since, in general, the scalar γ_1 can not force $\mathbf{v} = 0$, $a_2^1 = a_2^3$ and $a_0^3 = 0 = a_1^3$ must hold in order for (22) to hold.

Now consider a vertex of valence $n \neq 4$ so that the local, unbiased choice of the tangent directions (18) implies $\alpha \neq 0$ at the vertex. By Corollary 2, the segment emanating from a vertex has $\deg(\alpha) \leq 1$. Consider the segment $\mathbf{b}^1(u,0)$ with α^1 *not linear* closest to the vertex in the sense that its neighbor segment has α^3 linear. Then $a_0^3 = a_1^3 = 0$ implies $\alpha^3 \equiv 0$. Applying (10) and (13) repeatedly to the linear $\alpha_{\cdot,j}$ when approaching the vertex, propagates $\alpha_{\cdot,j} \equiv 0$. This is incompatible with $\alpha \neq 0$ at the vertex, completing the proof. |||

3.3 One double internal knot

Due to the vertex-local construction and since the double (not triple) knot implies C^1 continuity, the two-piece cubic boundary curve common to the two spline patches is completely determined. Moreover, the values of α are determined at the endpoints of the curve by the unbiased choice of the tangent directions (18), page 7.

Lemma 8 (one double interior knot) *In general, no vertex-localized construction enforcing unbiased G^1 constraints is possible, using bi-3 splines with one internal double knot.*

Proof let $(0, 0)$ corresponding to the breakpoint of the boundary curve. Here Lemma 2 applies and all vectors in (11) are fixed. By Corollary 2, α^1 and α^3 must both be linear. The claim follows since, in general, the only remaining single free *scalar* $\alpha^1(0) =: a^1_0 = -\alpha^3(0)$ cannot always enforce the vector-valued constraint (11). |||

4 Discussion and Conclusion

Using the Justification (page 3) and the assumption that all constructions are vertex-local, we could derive lower bounds from unbiased, logically symmetric constructions. Conversely, the construction in [FP08] establishes a tight upper bound on the number of polynomial pieces: It uses double internal knots and does not have the shape problem characterized by Lemma 5.

It has long been know that if all valences are odd or tangents are in an X configuration, vertex-enclosure does not impose constraints and simple Bézier constructions are possible (e.g. [Pet91, GZ94]). More global constructions, singular parameterization, or restrictions on the valence, for example by splitting patches, can allow for simpler constructions, e.g. [Rei95], [PBP02, 9.11], [Pet91, Pet95b]. We note also that $n^0 = n^1$ then we can choose linear α^1 and α^3 with $a^1_1 = a^3_1$ and $a^1_0 = 0$ to enforce (11) and arrive at a 2x2-split construction that is ruled out in general by Lemma 8.

Lemma 4 prevents a vertex-local solution with all α_{0j} linear. When the lemma is specialized by fixing the degree to be 3, increasing the patch continuity to C^2 and choosing $\alpha_{0j} := \frac{q-j}{q}\alpha_{00} + \frac{j}{q}\alpha_{0q}$ then we arrive at the statement (but not the proof sketch) of [SWWL04, Thm 3.1]. So, if we insist on single internal knots everywhere, a subdivision-like construction as proposed in [SWWL04] is justified by Lemma 4.

The results provide a checklist for constructions that are otherwise difficult to verify. Lemma 8 for example indicates that there must be a subtle error in the construction [HBC08] although, as mentioned above, such 2x2-split construction can succeed in special cases, such as smoothing a cube.

Constructions of smooth surfaces with one patch per quad are shown possible for degree bi-5, for example by [MYP08]. For degree bi-4, a single knot (a 2x2-split) must be introduced (see e.g. [Pet95a]).

Remarkably, the results in Section 2 do not depend on the degree or even the polynomial nature of splines, but assume only sufficiently smooth functions, that are piecewise with smooth transitions between the pieces.

References

[Bez77] Pierre E. Bezier. *Essai de definition numerique des courbes et des surfaces experimentales*. Ph.d. thesis, Universite Pierre et Marie Curie, February 1977.

[CC78] E. Catmull and J. Clark. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer Aided Design*, 10:350–355, 1978.

[FP08] Jianhua Fan and Jörg Peters. On smooth bicubic surfaces from quad meshes. In G. Bebis et al., editor, *ISVC (1)*, volume 5358 of *Lecture Notes in Computer Science*, pages 87–96. Springer, 2008.

[GZ94] John A. Gregory and Jianwei Zhou. Filling polygonal holes with bicubic patches. *Computer Aided Geometric Design*, 11(4):391–410, 1994. ISSN 0167-8396.

[HBC08] Stefanie Hahmann, Georges-Pierre Bonneau, and Baptiste Caramiaux. Bicubic G1 interpolation of irregular quad meshes using a 4-split. In Falai Chen and Bert Jüttler, editors, *Advances in Geometric Modeling and Processing, 5th International Conference, GMP 2008, Hangzhou, China, April 23-25, 2008. Proceedings*, volume 4975 of *Lecture Notes in Computer Science*, pages 17–32. Springer, 2008.

[LS08] Charles Loop and Scott Schaefer. Approximating Catmull-Clark subdivision surfaces with bicubic patches. *ACM Trans. Graph.*, 27(1):1–11, 2008.

[MKP07] Ashish Myles, Kestutis Karčiauskas, and Jörg Peters. Extending Catmull-Clark subdivision and PCCM with polar structures. In *PG '07: Proceedings of the 15th Pacific Conference on Computer Graphics and Applications*, pages 313–320, Washington, DC, USA, 2007. IEEE Computer Society.

[MYP08] A. Myles, Y. Yeo, and J. Peters. GPU conversion of quad meshes to smooth surfaces. In D. Manocha et al, editor, *ACM SPM*, pages 321–326, 2008.

[NYM+08] T. Ni, Y. Yeo, A. Myles, V. Goel, and J. Peters. GPU smoothing of quad meshes. In M. Spagnuolo et al, editor, *IEEE SMI*, pages 3–10, 2008.

[PBP02] H. Prautzsch, W. Boehm, and M. Paluzny. *Bézier and B-Spline Techniques*. Springer Verlag, 2002.

[Pet91] J. Peters. Smooth interpolation of a mesh of curves. *Constructive Approximation*, 7:221–247, 1991. Winner of SIAM Student Paper Competition 1989.

[Pet94] J. Peters. A characterization of connecting maps as roots of the identity. *Curves and Surfaces in Geometric Design*, pages 369–376, 1994.

[Pet95a] J. Peters. Biquartic C^1 spline surfaces over irregular meshes. *Computer Aided Design*, 27(12):895–903, December 1995.

[Pet95b] J. Peters. C^1-surface splines. *SIAM Journal on Numerical Analysis*, 32(2):645–666, 1995.

[Pet00] J. Peters. Patching Catmull-Clark meshes. In K. Akeley, editor, *ACM Siggraph*, pages 255–258, 2000.

[Pet01a] J. Peters. Modifications of PCCM. TR 001, Dept CISE, U Fl, 2001.

[Pet01b] J. Peters. Modifications of PCCM. Technical Report 2001-001, Dept CISE, University of Florida, 2001.

[Pet02] J. Peters. Geometric continuity. In *Handbook of Computer Aided Geometric Design*, pages 193–229. Elsevier, 2002.

[Rei95] Ulrich Reif. Biquadratic G-spline surfaces. *Computer Aided Geometric Design*, 12(2):193–205, 1995.

[SWWL04] Xiquan Shi, Tianjun Wang, Peiru Wu, and Fengshan Liu. Reconstruction of convergent G1 smooth B-spline surfaces. *Computer Aided Geometric Design*, pages 893–913, 2004.

[VPBM01] Alex Vlachos, Jorg Peters, Chas Boyd, and Jason L. Mitchell. Curved PN triangles. In *2001, Symposium on Interactive 3D Graphics*, Bi-Annual Conference Series, pages 159–166. ACM Press, 2001.

[vW86] J. van Wijk. Bicubic patches for approximating non-rectangular control-point meshes. *Computer Aided Geometric Design*, 3(1):1–13, 1986.