THE DISTANCE BETWEEN A UNIFORM (B–)SPLINE AND ITS CONTROL POLYGON

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ABSTRACT. We prove a sharp bound on the distance between a uniform spline and its B–Spline control polygon in terms of the second differences of the control points. This bound decreases by a factor of 4 under uniform refinement.

1. UNIFORM B–SPLINES

A piecewise polynomial p of degree d in B–Spline form,

$$p(t) = \sum_{k\in\mathbb{Z}} b_k N_k^d(t)$$

is defined by a nondecreasing *knot sequence* (t_k) and the *control points* (b_k). Both (t_k) and (b_k) are sequences of real numbers which we assume, for simplicity, to be biinfinite. The knot sequence enters this equation through the well–known recursion formulas for the B–Spline basis, see e.g. [dB93].

For *uniform* B–Splines, the knots are all equidistant and we may choose the knot sequence $t_k = k$ without loss of generality .

The *control polygon* ℓ of p is the piecewise linear interpolant of the control points b_k attached to their *Greville abscissae* t_k^*,

$$t_k^* = \frac{1}{d}\sum_{i=k+1}^{k+d} i = k + \frac{d+1}{2}.$$

Over the interval $[t_k^*, t_{k+1}^*]$ the k–th piece ℓ_k of the control polygon ℓ is given by

$$\ell_k(t) = b_k(t_{k+1}^* - t) + b_{k+1}(t - t_k^*) \qquad t \in [t_k^*, t_{k+1}^*].$$

The centered second differences of b are defined as $\Delta_2 b_k = b_{k-1} - 2b_k + b_{k+1}$.

With this notation in place, we can state our main result:

Theorem 1.

$$\|p - \ell\| \le \frac{d+1}{24}\|\Delta_2 b\|, \tag{1.1}$$

where $\|p-\ell\| = \max_{t\in\mathbb{R}} |p(t) - \ell(t)|$ *and* $\|\Delta_2 b\| = \max_{k\in\mathbb{Z}} |\Delta_2 b_k|$.

The proof of the theorem is along the same lines as the proof of the corresponding inequality for polynomials in Bézier–form in [NPL98].

Proof. We will prove the theorem by concentrating on each linear piece of the control polygon separately and showing that, for any k, $\|p - \ell_k\| \le \|\Delta_2 b\|\,(d +$

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1)/24 over $[t_k^*, t_{k+1}^*]$. Once we have established that, it follows that $\|p-\ell\| = \max_k \|p-\ell_k\|$ is also bounded by $\|\Delta_2 b\|\,(d+1)/24$.

We write $p-\ell$ over $[t_k^*, t_{k+1}^*]$ as $\sum_i b_i\,\alpha_{ki}(t)$, where

$$\alpha_{ki}(t) = N_i^d(t) - \begin{cases} t_{k+1}^* - t & i = k \\ t - t_k^* & i = k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Only finitely many α_{ki} have support on $[t_k^*, t_{k+1}^*]$. Which ones exactly depends on the degree d and on k, but not on i. For simplicity let $f = f(d,k)$ and $g = g(d,k)$ be such that $\alpha_{ki} = 0$ for any $t \in [t_k^*, t_{k+1}^*]$ and all $i \notin \{f, f+1, \ldots, g\}$.

The partition of unity $\sum_i N_i^d = 1$ and the fact that $t_{k+1}^* - t_k^* = 1$ imply that $\sum_i \alpha_{ki} = 0$. Similarly,

$$\sum_i i\,\alpha_{ki} = \sum_i t_i^* \alpha_{ki} - \sum_i \frac{d+1}{2}\alpha_{ki} = t - t_k^*(t_{k+1}^* - t) - t_{k+1}^*(t - t_k^*) - 0 = 0$$

because of the linear precision of B–Splines, $\sum_i t_i^* N_i^d = t$.

These two equalities together prove that the functions β_{ki} given by

$$\beta_{ki}(t) = \sum_{j=-\infty}^{i} (i-j)\alpha_{kj}$$

are all non-negative: since $\sum_i \alpha_{ki} = \sum_i i\alpha_{ki} = 0$ we can express β_{ki} as

$$\beta_{ki}(t) = \begin{cases} \sum_{j=-\infty}^{i} (i-j) N_j^d & i \le k \\ \sum_{j=i}^{\infty} (j-i) N_j^d & i \ge k+1. \end{cases}$$

The non-negativity of the β_{ki} now follows from the fact that all the coefficients on the right hand side and all the N_j^d are non-negative.

Again, only finitely many of the β_{ki} are non–zero. More specifically, we have $\beta_{ki} = 0$ for $i < f$ since $\alpha_{ki} = 0$ for these values of i; we also have that $\beta_{ki} = 0$ for $i > g$ since $\sum_{j=f}^{g} (i-j)\alpha_{kj} = 0$.

The β are second anti–differences of the α, in the sense that $\alpha_{ki} = \Delta_2 \beta_{ki}$. Thus, we can use summation by parts to rewrite $p - \ell_k$ through

$$p - \ell_k = \sum_i b_i\,\alpha_{ki} = \sum_i b_i\,\Delta_2 \beta_{ki} = \sum_i \Delta_2 b_i\,\beta_{ki}$$

and arrive over the interval $[t_k^*, t_{k+1}^*]$ at the estimate

$$\|p - \ell_k\| = \|\sum_i \Delta_2 b_i\,\beta_{ki}\| \le \|\Delta_2 b\|\,\|\sum_i \beta_{ki}\| = \|\Delta_2 b\|\,\sum_i \beta_{ki}.$$

The theorem follows if we can show that $q(t) := \sum_i \beta_{ki} \le (d+1)/24$.

$$\begin{aligned} \sum_i \beta_{ki} = \sum_{i=f}^{g} \beta_{ki} = \sum_{i=f}^{g}\sum_{j=f}^{i} (i-j)\alpha_{kj} = \sum_{j=f}^{g}\sum_{i=j}^{g} (i-j)\alpha_{kj} = \sum_j \alpha_{kj} \sum_{i=0}^{g-j} i \\ = \sum_j \binom{j-g}{2} \alpha_{kj} = \sum_j \binom{j}{2} \alpha_{kj} = \sum_j \binom{j-k}{2} N_j^d. \end{aligned}$$

$\sum_j \binom{j-k}{2} N_j^d$ is a quadratic polynomial which we express in monomial form using the formulas in appendix A:

$$\begin{aligned}\sum_i \beta_{ki} &= t^2/2 - (k + \frac{d+2}{2})t + \frac{1}{8}(d+2k)^2 + \frac{1}{24}(24k + 13d + 10) \\ &= \frac{1}{2}\left(t^2 - (t_k^* + t_{k+1}^*)t + \left(\frac{t_k^* + t_{k+1}^*}{2}\right)^2 + \frac{d-2}{12}\right).\end{aligned}$$

This shows that q is a non-negative quadratic polynomial with positive leading coefficient. It therefore attains its maximum over $[t_k^*, t_{k+1}^*]$ at one of the endpoints of the interval. Its values there are

$$q(t_k^*) = q(t_{k+1}^*) = \frac{d+1}{24}$$

and hence $q(t) \le (d+1)/24$ for all $t \in [t_k^*, t_{k+1}^*]$. □

2. Uniform refinement

An important operation on B–Splines is the refinement of the knot sequence or knot insertion. It changes the representation of the piecewise polynomial p over the old knot sequence to one over an enlarged knot sequence. As a consequence, the new control polygon $\hat{b}$ approximates p more closely than the old control polygon b.

For uniform B–Splines, we consider the refinement of the knot sequence $t_k = k$ to the sequence $\hat{t}_k = k/2$, halving the distance between knots. We now have two representations for p,

$$p(t) = \sum_k b_k N^d(t-k) = \sum_k \hat{b}_k N^d(2t-k),$$

where new control points $\hat{b}_k$ are given by

$$\hat{b}_{2i} = 2^{-d} \sum_{j=0}^{\lceil d/2 \rceil} \left(\binom{d}{2j-1} + \binom{d}{2j} \right) b_{i-j} = 2^{-d} \sum_{j=0}^{\lceil d/2 \rceil} \binom{d+1}{2j} b_{i-j}, \tag{2.2}$$

$$\hat{b}_{2i+1} = 2^{-d} \sum_{j=0}^{\lceil d/2 \rceil} \left(\binom{d}{2j} + \binom{d}{2j+1} \right) b_{i-j} = 2^{-d} \sum_{j=0}^{\lceil d/2 \rceil} \binom{d+1}{2j+1} b_{i-j}. \tag{2.3}$$

Lemma 2. *The second differences $\Delta_2 \hat{b}_i$ of the new control polygon are related to the second differences $\Delta_2 b_i$ of the old control polygon through*

$$2^d \Delta_2 \hat{b}_{2i} = \sum_j \binom{d-1}{2j-1} \Delta_2 b_{i-j}$$

$$2^d \Delta_2 \hat{b}_{2i+1} = \sum_j \binom{d-1}{2j} \Delta_2 b_{i-j}.$$

Proof. We use the fact that the second differences Δb_i and $\Delta \hat{b}_i$ are related to the second derivative of p by

$$\begin{aligned} p''(t) &= \sum_k \nabla b_k N^{d-2}(t-k) = \sum_k \nabla \hat{b}_k N^{d-2}(2t-k) \\ &= \sum_k \Delta b_{k-1} N^{d-2}(t-k) = \sum_k \Delta \hat{b}_{k-1} N^{d-2}(2t-k). \end{aligned}$$

We have to be careful to change from the backward second differences $\nabla a_i = a_i - 2a_{i-1} + a_{i-2}$ to the centered second differences Δ. They are related by $\nabla a_i = \Delta a_{i-1}$.

Together with the refinement formulas (2.2) and (2.3) we get for even indices

$$\nabla \hat{b}_{2i+2} = 2^{-d} \sum_{j=0}^{\lceil \frac{d-2}{2} \rceil} \binom{d-1}{2j} \nabla b_{i+1-j}$$

$$\Delta \hat{b}_{2i+1} = 2^{-d} \sum_{j=0}^{\lceil d/2 \rceil - 1} \binom{d-1}{2j} \Delta b_{i-j}$$

and for the odd indices

$$\nabla \hat{b}_{2i+1} = 2^{-d} \sum_{j=0}^{\lceil \frac{d-2}{2} \rceil} \binom{d-1}{2j+1} \nabla b_{i-j}$$

$$\Delta \hat{b}_{2i} = 2^{-d} \sum_{j=0}^{\lceil d/2 \rceil - 1} \binom{d-1}{2j+1} \nabla b_{i-(j+1)}$$

$$= 2^{-d} \sum_{j=1}^{\lceil d/2 \rceil} \binom{d-1}{2j-1} \Delta b_{i-j}$$

□

Corollary 3.

$$\|\Delta_2 \hat{b}\| = \frac{1}{4} \|\Delta_2 b\|.$$

Proof. This is a direct consequence of the fact that $\sum_j \binom{d-1}{j} = 2^{d-1}$ and the symmetry of the binomial coefficients, which imply that

$$\sum_j \binom{d-1}{2j} = \sum_j \binom{d-1}{2j-1} = 2^{d-2}.$$

□

2.1. **Special cases.** For quadratic B–Splines, uniform refinement is Chaikin's algorithm and

$$\hat{b}_{2i} = 2^{-2}(3b_{i-1} + b_i)$$

$$\hat{b}_{2i+1} = 2^{-2}(b_{i-1} + 3b_i).$$

This yields

$$\Delta_2 \hat{b}_{2i} = \Delta_2 \hat{b}_{2i-1} = \frac{1}{4} \Delta_2 b_{i-1}$$

.

Similarly, for cubic B–Splines we have

$$\hat{b}_{2i} = 2^{-3}(b_{i-2} + 6b_{i-1} + b_i)$$

$$\hat{b}_{2i+1} = 2^{-3}(4b_{i-1} + 4b_i).$$

and

$$\Delta_2 \hat{b}_{2i} = \frac{1}{4} \Delta_2 b_{i-1} \quad \text{and} \quad \Delta_2 \hat{b}_{2i+1} = \frac{1}{8}(\Delta_2 b_{i-1} + \Delta_2 b_i).$$

Appendix A. Representing t^2 as a uniform B–Spline

In general, monomials can be represented as B–Splines through[1]

$$\binom{d}{j} t^j = \sum_k \frac{1}{d!} \left(\sum_{\sigma \in \Pi_d} \prod_{i=1}^{j} t_{k+\sigma(i)} \right) N_k^d$$

Here, Π_d is the set of all permutations of the numbers $\{1, \ldots, d\}$.

For uniform B–Splines and $j = 2$, this boils down to:

$$\begin{aligned}
\binom{d}{2} t^2 &= \frac{1}{2} \sum_k \sum_{\sigma \in \Pi_d} (t_{k+\sigma(1)} t_{k+\sigma(2)}) N_k^d \\
&= \sum_k \sum_{\sigma \in \Pi_d} ((k+\sigma(1))(k+\sigma(2))) N_k^d \\
&= \frac{1}{2} \sum_k \left(\sum_{p=1}^{d} \sum_{q=1}^{d} (k+p)(k+q) - \sum_{p=1}^{d} (k+p)^2 \right) N_k^d \\
&= \frac{1}{2} \sum_k \left[\frac{d^2}{4} (2k+d+1)^2 - \frac{d}{6} (6k(k+d+1) + 2d^2 + 3d + 1) \right] N_k^d \\
&= \sum_k \frac{1}{12} \binom{d}{2} (3(d+2k)^2 + 5d + 2 + 12k) N_k^d.
\end{aligned}$$

References

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[1] This is a corrected version of a formula given in [Gol95]. The original formula is missing the factor of $1/d!$ on the right hand side.