Generalized Capacitance Matrix Theorems for Solving Linear Systems[1]

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Abstract

The capacitance matrix method has been widely used as an efficient numerical tool for solving the boundary value problems on irregular regions. Initially, this method was based on the Sherman-Morrison-Woodbury formula, an expression for the inverse of the matrix $(\mathbf{A}+\mathbf{U}\mathbf{V}^T)$ with $\mathbf{A} \in \Re^{n\times n}$ and $\mathbf{U}, \mathbf{V} \in \Re^{n\times p}$. Extensions to the capacitance matrix method with restrictive assumptions on the matrices $\mathbf{A}$ and $(\mathbf{A}+\mathbf{U}\mathbf{V}^T)$ have been reported in the past. In this paper, we present several theorems which are generalizations of the capacitance matrix theorem [2] and are suited for very general matrices $\mathbf{A}$ and $(\mathbf{A}+\mathbf{U}\mathbf{V}^T)$. A modified capacitance matrix method is developed from these theorems and holds the promise of being applicable to more general boundary value problems; in addition, it gives ample freedom to choose the matrix $\mathbf{A}$ for developing very efficient numerical algorithms.

1 Introduction

There are numerous applications of the Sherman-Morrison-Woodbury formula in various fields [5][14]. These applications can be classified into two categories. One is to compute the inverse of a matrix that has low-rank difference w.r.t. another matrix whose inverse is available or can be efficiently computed; the other is to solve a linear system whose matrix differs from a well-structured matrix by a low-rank matrix and very efficient methods exist to solve the linear system with the well-structured matrix. The latter case is also called the capacitance matrix method. This method has been widely used for solving linear systems arising from the discretization of the boundary value problems, primarily for the elliptic partial differential equations. *In this paper, we state and prove some very general theorems which are then applied in developing a new capacitance matrix method.*

When $\mathbf{A} \in \Re^{n\times n}$, $\mathbf{U}, \mathbf{V} \in \Re^{n\times p}$, and both the matrices $\mathbf{A}$ and $(I+\mathbf{V}^T\mathbf{A}^{-1}\mathbf{U})$ are nonsingular, the Sherman-Morrison-Woodbury formula [4] gives an expression for the inverse of $(\mathbf{A}+\mathbf{U}\mathbf{V}^T)$ as

$$(\mathbf{A}+\mathbf{U}\mathbf{V}^T)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(I+\mathbf{V}^T\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}^T\mathbf{A}^{-1}. \tag{1}$$

The above assumptions imply that the matrix $(\mathbf{A}+\mathbf{U}\mathbf{V}^T)$ is also nonsingular. The matrix $(I+\mathbf{V}^T\mathbf{A}^{-1}\mathbf{U})$ is called the capacitance matrix and is denoted by $\mathbf{C}$. Note that a similar formula can be obtained for the inverse of $(\mathbf{A}+\mathbf{U}\mathbf{G}\mathbf{V}^T)$, where the matrix $\mathbf{G} \in \Re^{p\times p}$ is nonsingular [5].

Let $\mathbf{B} = \mathbf{A}+\mathbf{U}\mathbf{V}^T$, with $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ being nonsingular, then the linear system $\mathbf{B}\mathbf{x} = \mathbf{b}$ can be solved using the traditional capacitance matrix method, which involves the following steps:

1. Solve $\mathbf{A}\bar{\mathbf{x}} = \mathbf{b}$ for $\bar{\mathbf{x}}$.
2. Compute $\mathbf{W} = \mathbf{A}^{-1}\mathbf{U}$ by solving $\mathbf{A}\mathbf{W} = \mathbf{U}$ column by column.
3. Form the capacitance matrix $\mathbf{C} = I + \mathbf{V}^T\mathbf{W}$.
4. Solve $\mathbf{C}\boldsymbol{\beta} = \mathbf{V}^T\bar{\mathbf{x}}$ for $\boldsymbol{\beta}$.

5. Form/Compute the solution $\mathbf{x} = \bar{\mathbf{x}} - \mathbf{W}\boldsymbol{\beta}$.

This capacitance matrix method is derived directly from the Sherman-Morrison-Woodbury formula, and therefore, it can be applied only when $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are all nonsingular, making it unsuitable for general linear system problems. In addition, the choice of a well-structured matrix $\mathbf{A}$ which is close to the original matrix is limited to be nonsingular.

Buzbee et al. [2] proposed a fast algorithm to solve the Poisson equation (a second-order elliptic PDE) on irregular regions. Their algorithm used a domain imbedding technique to imbed the irregular region in a rectangular region and utilized a fast Poisson solver (on the rectangular region) which is based on the capacitance matrix method. The discrete Poisson operator on the rectangular domain is the Laplacian matrix, which is singular with rank $n-1$. Therefore, they presented a theorem to generalize the traditional capacitance matrix method to the case when $rank(\mathbf{A}) = n-1$ and $\mathbf{B}$ is nonsingular. This generalization is based on modifying the right-hand-sides (RHSs) of the linear systems appearing in the first and second steps.

The capacitance matrix method has been further employed as a discrete analogy to the potential theory for partial differential equations by Proskurowski and Wildlund [9] [10], Shieh [12] [13], O'Leary and Wildlund [8], Dryja [3], Borgers and Wildlund [1], and others. They used the framework of the capacitance matrix method to solve the second-order elliptic boundary value problems. Although the matrix $\mathbf{B}$ could be singular for the Neumann problem, the chosen matrix $\mathbf{A}$ was assumed to be nonsingular. This limits the choice of the well-structured matrix $\mathbf{A}$ in designing a more efficient algorithm.

In this paper, we present several theorems which we call generalized capacitance matrix theorems that lead to a modified capacitance matrix method which can deal with very general linear systems. In this method, there are no restrictions on the rank of the matrices $\mathbf{A}$ or $\mathbf{B}$ and it can be applied to solve any linear system as long as it has a solution. In addition, with this generalized capacitance matrix method, we have more freedom to choose the well-structured matrix $\mathbf{A}$.

In the next section, we will present the generalized capacitance matrix theorems for different types of relationships between the matrices $\mathbf{A}$ and $\mathbf{B}$. Some discussion regarding how these theorems may be used to modify the traditional capacitance matrix method will be given in the final section.

2 Generalized Capacitance Matrix Theorems

The generalized capacitance matrix theorems are distinguished according to the types of relationship between the matrices $\mathbf{A}$ and $\mathbf{B}$. The first theorem is for the case when $null(\mathbf{B}) \subset null(\mathbf{A})$ and $range(\mathbf{A}) \subset range(\mathbf{B})$, while the second theorem is for the assumption $null(\mathbf{A}) \subset null(\mathbf{B})$ and $range(\mathbf{B}) \subset range(\mathbf{A})$. There are no assumptions on $\mathbf{A}$ and $\mathbf{B}$ in the third theorem. The generalized capacitance matrix theorem for the first case is given as follows.

Theorem 1 *Given* $\mathbf{B} = \mathbf{A} + \mathbf{U}\mathbf{G}\mathbf{V}^T$, *where* $\mathbf{A}, \mathbf{B} \in \Re^{n\times n}$, $\mathbf{G} \in \Re^{p\times p}$ *is nonsingular,* $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_p] \in \Re^{n\times p}$, $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_p] \in \Re^{n\times p}$, *with both the sets* $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ *and* $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *containing linearly independent vectors. Assume* $null(\mathbf{B}) \subset null(\mathbf{A})$ *and* $range(\mathbf{A}) \subset range(\mathbf{B})$. *Let* $rank(\mathbf{B}) = n - m_B$ *and* $rank(\mathbf{A}) = n - m_A$, $0 \le m_B \le m_A \le n$, *then there exist two sets of orthogonal basis,* $\{\mathbf{q}_1, \ldots, \mathbf{q}_{m_A}\}$ *and* $\{\mathbf{q'}_1, \ldots, \mathbf{q'}_{m_A}\}$, *for* $null(\mathbf{A}^T)$ *and* $null(\mathbf{A})$ *respectively, such that* $null(\mathbf{B}^T) = span\{\mathbf{q}_{m_A-m_B+1}, \ldots, \mathbf{q}_{m_A}\}$ *and* $null(\mathbf{B}) = span\{\mathbf{q'}_{m_A-m_B+1}, \ldots, \mathbf{q'}_{m_A}\}$. *Let* $\mathbf{q}_i^T\mathbf{u}_j = \alpha_i\delta(i,j)$, $\alpha_i \neq 0$, *for* $1 \le i, j \le m_A - m_B$.

Let $\bar{\mathbf{x}}$ *be a solution to*

$$\mathbf{A}\bar{\mathbf{x}} = \mathbf{b} - \sum_{j=1}^{m_A-m_B} \frac{\mathbf{q}_j^T\mathbf{b}}{\mathbf{q}_j^T\mathbf{u}_j}\mathbf{u}_j, \tag{2}$$

where $\mathbf{b} \in range(\mathbf{B})$. *For* $1 \le i \le p$, *let* $\boldsymbol{\eta}_i$ *be a solution to*

$$\mathbf{A}\boldsymbol{\eta}_i = \mathbf{u}_i - \sum_{j=1}^{m_A-m_B} \frac{\mathbf{q}_j^T\mathbf{u}_i}{\mathbf{q}_j^T\mathbf{u}_j}\mathbf{u}_j, \tag{3}$$

and choose $\boldsymbol{\eta}_1 \ldots \boldsymbol{\eta}_{m_A-m_B}$ *to be linear combinations of* $\mathbf{q'}_1, \ldots, \mathbf{q'}_{m_A-m_B}$ *and linearly independent. Define the capacitance matrix* $\mathbf{C}$ *as*

$$\mathbf{C} = \mathbf{G}^{-1} + \mathbf{V}^{\mathbf{T}}\boldsymbol{\eta} - \sum_{j=1}^{m_A-m_B} \mathbf{G}^{-1}\mathbf{e}_j\frac{\mathbf{q}_j^T\mathbf{U}}{\mathbf{q}_j^T\mathbf{u}_j}, \tag{4}$$

where $\boldsymbol{\eta} = [\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_p]$, *and* $\mathbf{e}_j$ *is the j-th standard basis vector in* $\Re^p$. *Then there exists a unique solution* $\boldsymbol{\beta}$ *to the capacitance matrix equation*

$$\mathbf{C}\boldsymbol{\beta} = \mathbf{V}^T\bar{\mathbf{x}} - \sum_{j=1}^{m_A-m_B} \frac{\mathbf{q}_j^T\mathbf{b}}{\mathbf{q}_j^T\mathbf{u}_j}\mathbf{G}^{-1}\mathbf{e}_j, \tag{5}$$

and $\bar{\mathbf{x}} - \boldsymbol{\eta}\boldsymbol{\beta}$ *is a solution to the linear system* $\mathbf{B}\mathbf{x} = \mathbf{b}$.

We will use the following proposition to prove this theorem.

Proposition 1 *Under the assumptions in Theorem 1, we have* $\mathbf{v}_i^T\mathbf{q'}_j = \mathbf{u}_i^T\mathbf{q}_j = 0$ *for* $1 \le i \le p$ *and* $m_A - m_B + 1 \le j \le m_A$. *In addition,* $\mathbf{V}^T[\mathbf{q'}_1 \ldots \mathbf{q'}_{m_A-m_B}]$ *and* $\mathbf{U}^T[\mathbf{q}_1 \ldots \mathbf{q}_{m_A-m_B}]$ *are of full rank.*

Proof: From the assumption in Theorem 1, $\mathbf{B}\mathbf{q'}_j = \mathbf{A}\mathbf{q'}_j = \mathbf{0}$, $m_A - m_B + 1 \le j \le m_A$. Since $\mathbf{B}\mathbf{q'}_j = (\mathbf{A} + \mathbf{U}\mathbf{G}\mathbf{V}^T)\mathbf{q'}_j$, we have $\mathbf{U}\mathbf{G}\mathbf{V}^T\mathbf{q'}_j = \mathbf{0}$ for $m_A - m_B + 1 \le j \le m_A$. From the assumption that $\mathbf{U} \in \Re^{n\times p}, p \le n$, has full rank and $\mathbf{G}$ is nonsingular, it is obvious that $\mathbf{V}^T\mathbf{q'}_j = \mathbf{0}$. Therefore, we have $\mathbf{v}_i^T\mathbf{q'}_j = 0$ for $1 \le i \le p$ and $m_A - m_B + 1 \le j \le m_A$. Similarly, we can show that $\mathbf{u}_i^T\mathbf{q}_j = 0$, $1 \le i \le p$ and $m_A - m_B + 1 \le j \le m_A$.

We use proof by contradiction to show that the matrix $\mathbf{V}^T[\mathbf{q'}_1 \ldots \mathbf{q'}_{m_A-m_B}]$ is of full rank. Let's assume this matrix is rank deficient, i.e., there exists a nonzero vector $\mathbf{a}^T = (a_1, \ldots, a_{m_A-m_B})$ such

that $\mathbf{V}^T[\mathbf{q}'_1 \ldots \mathbf{q}'_{m_A-m_B}]\mathbf{a} = \mathbf{0}$. Let $\mathbf{r} = [\mathbf{q}'_1 \ldots \mathbf{q}'_{m_A-m_B}]\mathbf{a}$, then $\mathbf{r} \in null(\mathbf{A})$ and $\mathbf{r}$ is orthogonal to $null(\mathbf{B})$. Post-multiplying both sides of the equation $\mathbf{B} = \mathbf{A} + \mathbf{UGV}^T$ by $\mathbf{r}$ yields $\mathbf{Br} = \mathbf{Ar} + \mathbf{UGV}^T\mathbf{r}$. Since $\mathbf{Ar} = \mathbf{0}$ and $\mathbf{V}^T\mathbf{r} = \mathbf{0}$, we obtain $\mathbf{Br} = \mathbf{0}$. But, from the above, $\mathbf{r}$ is a nonzero vector orthogonal to $null(\mathbf{B})$, i.e. $\mathbf{Br} \neq \mathbf{0}$, leading to a contradiction. Consequently, the assumption that $\mathbf{V}^T[\mathbf{q}'_1 \ldots \mathbf{q}'_{m_A-m_B}]$ is rank deficient is not true, implying that $\mathbf{V}^T[\mathbf{q}'_1 \ldots \mathbf{q}'_{m_A-m_B}]$ is of full rank. Similarly, we can prove that $\mathbf{U}^T[\mathbf{q}_1 \ldots \mathbf{q}_{m_A-m_B}]$ is of full rank. ■

We are now equipped to prove Theorem 1. The proof is in four parts, and we will show that (1) there exist two sets of orthogonal basis, $\{\mathbf{q}_1, \ldots, \mathbf{q}_{m_A}\}$ and $\{\mathbf{q}'_1, \ldots, \mathbf{q}'_{m_A}\}$, for $null(\mathbf{A}^T)$ and $null(\mathbf{A})$ respectively, such that $null(\mathbf{B}^T) = span\{\mathbf{q}_{m_A-m_B+1}, \ldots, \mathbf{q}_{m_A}\}$ and $null(\mathbf{B}) = span\{\mathbf{q}'_{m_A-m_B+1}, \ldots, \mathbf{q}'_{m_A}\}$, (2) there exist solutions to equations 2 and 3, (3) the capacitance matrix $\mathbf{C}$ is nonsingular, implying that equation 5 has a unique solution, and (4) $\mathbf{B}(\bar{\mathbf{x}} - \boldsymbol{\eta\beta}) = \mathbf{b}$.

Proof for Theorem 1:
(1) Since $rank(\mathbf{A}) = n - m_A$ and $rank(\mathbf{B}) = n - m_B$, it is obvious from their singular value decompositions that there exist m_A orthogonal eigenvectors, $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{m_A}$, for $\mathbf{A}$ and m_B orthogonal eigenvectors, $\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{m_B}$, for $\mathbf{B}$ such that $null(\mathbf{A}) = span\{\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{m_A}\}$ and $null(\mathbf{B}) = span\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{m_B}\}$. From the assumption $null(\mathbf{B}) \subset null(\mathbf{A})$, we can write each $\boldsymbol{\psi}_i$, $i = 1, \ldots, m$, in terms of a linear combination of the basis vectors $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{m_A}$. Thus, we have

$$\begin{pmatrix} \boldsymbol{\psi}_1 \\ \vdots \\ \boldsymbol{\psi}_{m_B} \end{pmatrix} = \mathbf{T} \begin{pmatrix} \boldsymbol{\phi}_1 \\ \vdots \\ \boldsymbol{\phi}_{m_A} \end{pmatrix}, \tag{6}$$

where $\mathbf{T} \in \Re^{m_B \times m_A}$ is of full rank. By using the Householder or Givens transformation [4], we can obtain the factorization of $\mathbf{T}$ as follows:

$$\mathbf{T} = \begin{bmatrix} \mathbf{0} & \bar{\mathbf{T}} \end{bmatrix} \mathbf{Q}, \tag{7}$$

where $\mathbf{0}$ is an $m_B \times (m_A - m_B)$ zero matrix, $\bar{\mathbf{T}} \in \Re^{m_B \times m_B}$ is nonsingular, and $\mathbf{Q} \in \Re^{m_A \times m_A}$ is orthogonal. Define the vectors $\mathbf{q}'_1, \ldots, \mathbf{q}'_{m_A}$ by the following relation

$$\begin{pmatrix} \mathbf{q}'_1 \\ \vdots \\ \mathbf{q}'_{m_A} \end{pmatrix} = \mathbf{Q} \begin{pmatrix} \boldsymbol{\phi}_1 \\ \vdots \\ \boldsymbol{\phi}_{m_A} \end{pmatrix}. \tag{8}$$

We can see that $\{\mathbf{q}'_1, \ldots, \mathbf{q}'_{m_A}\}$ is a set of orthogonal basis for $null(\mathbf{A})$ and its subset $\{\mathbf{q}'_{m_A-m_B+1}, \ldots, \mathbf{q}'_{m_A}\}$ forms an orthogonal basis for $null(\mathbf{B})$. Thus, the existence of the basis $\{\mathbf{q}'_1, \ldots, \mathbf{q}'_{m_A}\}$ which satisfies the aforementioned requirements is proved. Similarly, we can prove that there exists an orthogonal basis $\{\mathbf{q}_1, \ldots, \mathbf{q}_{m_A}\}$ for $null(\mathbf{A}^T)$ and $null(\mathbf{B}^T) = span\{\mathbf{q}_{m_A-m_B+1}, \ldots, \mathbf{q}_{m_A}\}$ is satisfied.

(2) From Proposition 1, we have $\mathbf{u}_i^T\mathbf{q}_j = 0$ for $1 \leq i \leq p$ and $m_A - m_B + 1 \leq j \leq m_A$. The assumption $\mathbf{b} \in range(\mathbf{B})$ implies $\mathbf{b}^T\mathbf{q}_j = 0$ for $m_A - m_B + 1 \leq j \leq m_A$. Combining these facts and the orthogonality of the eigenvectors $\mathbf{q}_j, j = 1, \ldots, m_A$, we can see that the RHSs of equations 2 and 3 are orthogonal to $\mathbf{q}_j$ for $j = 1, \ldots, m_A$, thus, the RHSs are orthogonal to $null(\mathbf{A}^T)$. Consequently,

the RHSs of equations 2 and 3 are both in the range of $\mathbf{A}$, which means there always exist solutions to equations 2 and 3.

(3) To prove that $\mathbf{C}$ is nonsingular, we show that $\mathbf{C}\boldsymbol{\beta} = \mathbf{0} \Rightarrow \boldsymbol{\beta} = \mathbf{0}$. Let's suppose $\mathbf{C}\boldsymbol{\beta} = \mathbf{0}$, then, from equation 4, we have

$$\mathbf{V}^T\boldsymbol{\eta}\boldsymbol{\beta} = \sum_{j=1}^{m_A-m_B} \mathbf{G}^{-1}\mathbf{e}_j \frac{\mathbf{q}_j^T\mathbf{U}\boldsymbol{\beta}}{\mathbf{q}_j^T\mathbf{u}_j} - \mathbf{G}^{-1}\boldsymbol{\beta}. \tag{9}$$

By definition, $\mathbf{B}\boldsymbol{\eta}\boldsymbol{\beta} = \mathbf{A}\boldsymbol{\eta}\boldsymbol{\beta} + \mathbf{U}\mathbf{V}^T\boldsymbol{\eta}\boldsymbol{\beta}$. Substituting equations 3 and 9 into this equation, we get $\mathbf{B}\boldsymbol{\eta}\boldsymbol{\beta} = \mathbf{0}$. Thus, $\boldsymbol{\eta}\boldsymbol{\beta} \in null(\mathbf{B})$. Since $null(\mathbf{B}) \subset null(\mathbf{A})$, we have $\boldsymbol{\eta}\boldsymbol{\beta} \in null(\mathbf{A})$, i.e. $\mathbf{A}\boldsymbol{\eta}\boldsymbol{\beta} = \mathbf{0}$. Substituting $\mathbf{A}\boldsymbol{\eta}$ by equation 3 and denoting $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)$, we obtain

$$\sum_{i=1}^{p} \mathbf{u}_i\beta_i - \sum_{i=1}^{p} \beta_i \sum_{j=1}^{m_A-m_B} \frac{\mathbf{q}_j^T\mathbf{u}_i}{\mathbf{q}_j^T\mathbf{u}_j}\mathbf{u}_j = \mathbf{0}. \tag{10}$$

After rearranging, equation 10 becomes

$$\sum_{j=1}^{m_A-m_B} (\beta_j - \frac{\mathbf{q}_j^T\mathbf{U}\boldsymbol{\beta}}{\mathbf{q}_j^T\mathbf{u}_j})\mathbf{u}_j + \sum_{j=m_A-m_B+1}^{p} \beta_j\mathbf{u}_j = \mathbf{0}. \tag{11}$$

Since $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are linearly independent, equation 11 implies the following conditions for β_j

$$\beta_j = \frac{\mathbf{q}_j^T\mathbf{U}\boldsymbol{\beta}}{\mathbf{q}_j^T\mathbf{u}_j}, \quad j = 1, \ldots, m_A - m_B, \tag{12}$$

$$\beta_j = 0, \quad j = m_A - m_B + 1, \ldots, p. \tag{13}$$

Applying these conditions into the RHS of equation 9, we obtain

$$\mathbf{V}^T[\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_{m_A-m_B}] \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{m_A-m_B} \end{pmatrix} = \mathbf{0}. \tag{14}$$

For $i = 1, \ldots, m_A - m_B$, the RHS in equation 3 equals $\mathbf{0}$, thus, $\boldsymbol{\eta}_i \in null(\mathbf{A})$. ¿From the assumption that $\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_{m_A-m_B}$ are chosen to be linear combinations of $\mathbf{q}'_1, \ldots, \mathbf{q}'_{m_A-m_B}$ and linearly independent, we can write $[\boldsymbol{\eta}_1 \ \ldots \ \boldsymbol{\eta}_{m_A-m_B}]$ in terms of $[\mathbf{q}'_1 \ \ldots \ \mathbf{q}'_{m_A-m_B}]$ as follows:

$$[\boldsymbol{\eta}_1 \ \ldots \ \boldsymbol{\eta}_{m_A-m_B}] = [\mathbf{q}'_1 \ \ldots \ \mathbf{q}'_{m_A-m_B}]\mathbf{T}, \tag{15}$$

where $\mathbf{T} \in \Re^{(m_A-m_B)\times(m_A-m_B)}$ is nonsingular. By substituting equation 15 into equation 14, equation 14 becomes

$$\mathbf{V}^T[\mathbf{q}'_1 \ \ldots \ \mathbf{q}'_{m_A-m_B}]\mathbf{T} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{m_A-m_B} \end{pmatrix} = \mathbf{0}. \tag{16}$$

¿From Proposition 1, the matrix $\mathbf{V}^T[\mathbf{q}'_1 \ \ldots \ \mathbf{q}'_{m_A-m_B}]$ is of full rank and it is of size $p \times (m_A - m_B)$ with $p \geq m_A - m_B$, which implies

$$\mathbf{T}\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{m_A-m_B} \end{pmatrix} = \mathbf{0}. \tag{17}$$

Since $\mathbf{T}$ is nonsingular, this makes $\beta_i = 0, i = 1, \ldots, m_A - m_B$. Combining this with equation 13, we obtain $\boldsymbol{\beta} = \mathbf{0}$. Thus concluding that $\mathbf{C}\boldsymbol{\beta} = \mathbf{0}$ implies $\boldsymbol{\beta} = \mathbf{0}$, therefore, $\mathbf{C}$ is nonsingular.

(4) Now, we will show that $\mathbf{B}(\bar{\mathbf{x}} - \boldsymbol{\eta}\boldsymbol{\beta}) = \mathbf{b}$. Substituting $\mathbf{A} + \mathbf{U}\mathbf{G}\mathbf{V}^{\mathbf{T}}$ for $\mathbf{B}$ and expanding, we have

$$\mathbf{B}(\bar{\mathbf{x}} - \boldsymbol{\eta}\boldsymbol{\beta}) = \mathbf{A}\bar{\mathbf{x}} - \mathbf{A}\boldsymbol{\eta}\boldsymbol{\beta} + \mathbf{U}\mathbf{G}\mathbf{V}^T\bar{\mathbf{x}} - \mathbf{U}\mathbf{G}\mathbf{V}^T\boldsymbol{\eta}\boldsymbol{\beta}. \tag{18}$$

Rewriting $\mathbf{V}^T\boldsymbol{\eta}\boldsymbol{\beta}$ using equations 4 and 5 and some algebraic manipulation, we get

$$\mathbf{V}^T\boldsymbol{\eta}\boldsymbol{\beta} = \mathbf{V}^T\bar{\mathbf{x}} - \sum_{j=1}^{m_A-m_B} \frac{\mathbf{q}_j^T\mathbf{b}}{\mathbf{q}_j^T\mathbf{u}_j}\mathbf{G}^{-1}\mathbf{e}_j - \mathbf{G}^{-1}\boldsymbol{\beta} + \sum_{j=1}^{m_A-m_B} \mathbf{e}_j\frac{\mathbf{q}_j^T\mathbf{U}}{\mathbf{q}_j^T\mathbf{u}_j}\boldsymbol{\beta}. \tag{19}$$

Substituting for $\mathbf{A}\bar{\mathbf{x}}$, $\mathbf{A}\boldsymbol{\eta}$ and $\mathbf{V}^T\boldsymbol{\eta}\boldsymbol{\beta}$ from equations 2, 3 and 19 respectively, equation 18 leads to $\mathbf{B}(\bar{\mathbf{x}} - \boldsymbol{\eta}\boldsymbol{\beta}) = \mathbf{b}$. ■

In theorem 1, we assume that the matrices $\mathbf{U}$ and $\mathbf{V}$ contain linearly independent columns and $\mathbf{G}$ is nonsingular, which is equivalent to saying that $\mathbf{U}, \mathbf{V}$ and $\mathbf{G}$ are all of full rank, in addition, the condition $\mathbf{q}_i^T\mathbf{u}_j = \alpha_i\delta(i,j)$ must be satisfied for $1 \leq i, j \leq m_A - m_B$. In fact, we can prove that given any two matrices $\mathbf{A}, \mathbf{B} \in \Re^{n\times n}$, there always exist full-ranked matrices $\mathbf{U}, \mathbf{V} \in \Re^{n\times p}$ and $\mathbf{G} \in \Re^{p\times p}$ such that $\mathbf{U}\mathbf{G}\mathbf{V}^T = \mathbf{B} - \mathbf{A}$ and $p = rank(\mathbf{B} - \mathbf{A})$. This can be verified by taking the singular value decomposition of the matrix $\mathbf{B} - \mathbf{A}$ as $\bar{\mathbf{U}}\boldsymbol{\Sigma}\bar{\mathbf{V}}^T$, where both $\bar{\mathbf{U}}$ and $\bar{\mathbf{V}}$ are orthogonal matrices and $\boldsymbol{\Sigma}$ is a diagonal matrix with rank $p = rank(\mathbf{B} - \mathbf{A})$. We can reduce the size of the matrix $\boldsymbol{\Sigma}$ by retaining the p nonzero diagonal entries and taking out the remaining rows and columns of zeros. This reduced matrix $\mathbf{G} \in \Re^{p\times p}$ is nonsingular. In addition, we take out the columns in $\bar{\mathbf{U}}$ and $\bar{\mathbf{V}}$ corresponding the zero singular values in $\boldsymbol{\Sigma}$, then the reduced matrices $\mathbf{U}, \mathbf{V} \in \Re^{n\times p}$ are obtained. It is not difficult to see that $\mathbf{U}\mathbf{G}\mathbf{V}^T = \mathbf{B} - \mathbf{A}$ and the matrices $\mathbf{U}, \mathbf{V}$ and $\mathbf{G}$ are of full rank.

As for the condition $\mathbf{q}_i^T\mathbf{u}_j = \alpha_i\delta(i,j)$, $1 \leq i, j \leq m_A - m_B$, we can find such a $\mathbf{U}$ by transformation. If $\{\mathbf{U}, \mathbf{G}, \mathbf{V}\}$ satisfies $\mathbf{U}\mathbf{G}\mathbf{V}^T = \mathbf{B} - \mathbf{A}$ and each is of full rank, then the alternative set $\{\mathbf{U}\mathbf{Q}, \mathbf{Q}^{-1}\mathbf{G}, \mathbf{V}\}$ also satisfies these requirements when the matrix $\mathbf{Q} \in \Re^{p\times p}$ is nonsingular. From Proposition 1, $\mathbf{U}^T[\mathbf{q}_1 \ldots \mathbf{q}_{m_A-m_B}]$ has full rank. We can find a transformation matrix $\mathbf{Q}$ to make the upper $(m_A - m_B) \times (m_A - m_B)$ block of the transformed matrix $\mathbf{Q}^T\mathbf{U}^T[\mathbf{q}_1 \ldots \mathbf{q}_{m_A-m_B}]$ diagonal. Then, the condition $\mathbf{q}_i^T\mathbf{u}_j = \alpha_i\delta(i,j)$, $1 \leq i, j \leq m_A - m_B$, is satisfied.

There are infinite number of choices for the matrices $\mathbf{U}, \mathbf{V}$ and $\mathbf{G}$ to satisfy the aforementioned requirements. The above choice from the singular value decomposition of $\mathbf{B} - \mathbf{A}$ is just a particular one that leads to a diagonal $\mathbf{G}$ matrix. This makes the computation of $\mathbf{G}^{-1}$ in equation 4 trivial. The choice of $\mathbf{A}, \mathbf{U}, \mathbf{V}$ and $\mathbf{G}$ is very crucial for the efficiency of the numerical algorithms based on the capacitance matrix method to solve a linear system.

Now let's turn to the other case when $null(\mathbf{A}) \subset null(\mathbf{B})$ and $range(\mathbf{B}) \subset range(\mathbf{A})$. Note that these two constraints are equivalent to each other when both $\mathbf{B}$ and $\mathbf{A}$ are symmetric.

Theorem 2 *Given* $\mathbf{B} = \mathbf{A} + \mathbf{U}\mathbf{G}\mathbf{V}^T$, *where* $\mathbf{A}, \mathbf{B} \in \Re^{n\times n}$, $\mathbf{G} \in \Re^{p\times p}$ *is nonsingular,* $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_p] \in \Re^{n\times p}$, $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_p] \in \Re^{n\times p}$, *with both the sets* $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ *and* $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *containing linearly independent vectors. Assume* $null(\mathbf{A}) \subset null(\mathbf{B})$ *and* $range(\mathbf{B}) \subset range(\mathbf{A})$. *Let* $\bar{\mathbf{x}}$ *be a solution to*

$$\mathbf{A}\bar{\mathbf{x}} = \mathbf{b}, \tag{20}$$

where $\mathbf{b} \in range(\mathbf{B})$. *For* $1 \leq i \leq p$, *let* $\boldsymbol{\eta}_i$ *be a solution to*

$$\mathbf{A}\boldsymbol{\eta}_i = \mathbf{u}_i, \tag{21}$$

Define the capacitance matrix $\mathbf{C}$ *as* $\mathbf{G}^{-1} + \mathbf{V}^T\boldsymbol{\eta}$, *where* $\boldsymbol{\eta} = [\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_p]$. *Then there exist a solution* $\boldsymbol{\beta}$ *to the capacitance matrix equation*

$$C\boldsymbol{\beta} = \mathbf{V}^T\bar{\mathbf{x}}, \tag{22}$$

and $\bar{\mathbf{x}} - \boldsymbol{\eta}\boldsymbol{\beta}$ *is a solution to the linear system* $\mathbf{B}\mathbf{x} = \mathbf{b}$.

We will use the following proposition to prove the above theorem.

Proposition 2 *In Theorem 2, let* $\{\mathbf{q}_1, \ldots, \mathbf{q}_{m_A}\}$ *and* $\{\mathbf{q'}_1, \ldots, \mathbf{q'}_{m_A}\}$ *be the sets of orthogonal eigenvectors in* $null(\mathbf{A}^T)$ *and* $null(\mathbf{A})$ *respectively, where* $m_A = n - rank(\mathbf{A})$. *Then,* $\mathbf{q}_i^T\mathbf{u}_j = \mathbf{q'}_i^T\mathbf{v}_j = 0$, *for* $i = 1, \ldots, m_A$ *and* $j = 1, \ldots, p$.

Proof: From the assumption that $null(\mathbf{A}) \subset null(\mathbf{B})$, we can see that the eigenvectors $\mathbf{q'}_i$, $i = 1, \ldots, m_A$, are also in $null(\mathbf{B})$. Since $\mathbf{B}\mathbf{q'}_i = \mathbf{A}\mathbf{q'}_i + \mathbf{U}\mathbf{G}\mathbf{V}^T\mathbf{q'}_i$, we obtain

$$\mathbf{U}\mathbf{G}\mathbf{V}^T\mathbf{q'}_i = \mathbf{0}, \tag{23}$$

for $1 \leq i \leq m_A$. The matrix $\mathbf{U}$ is of size $n \times p$, $n \geq p$, and of full rank, consequently $\mathbf{G}\mathbf{V}^T\mathbf{q'}_i = \mathbf{0}$. From the assumption that $\mathbf{G}$ is nonsingular, we have $\mathbf{V}^T\mathbf{q'}_i = \mathbf{0}$. Therefore, $\mathbf{q'}_i^T\mathbf{v}_j = 0$ for $i = 1, \ldots, m_A$ and $j = 1, \ldots, p$.

Similarly, we obtain the equation $\mathbf{V}\mathbf{G}^T\mathbf{U}^T\mathbf{q}_i = \mathbf{0}$ from the condition $\mathbf{B}^T\mathbf{q}_i = \mathbf{A}^T\mathbf{q}_i = \mathbf{0}$ which is derived from the assumption $range(\mathbf{B}) \subset range(\mathbf{A})$. Using the same reasoning on this equation leads to the conclusion that $\mathbf{q}_i^T\mathbf{u}_j = 0$, $i = 1, \ldots, m_A$ and $j = 1, \ldots, p$. ■

We are now poised to prove Theorem 2.

Proof for Theorem 2 : The proof is presented in three parts namely, (1) we will show that there exist solutions for equations 20 and 21, (2) we will establish that there exist a solution for equation 22, and (3) show that $\mathbf{B}(\bar{\mathbf{x}} - \boldsymbol{\eta}\boldsymbol{\beta}) = \mathbf{b}$.

(1) To show that there exist a solution for equations 20 and 21, we prove that the RHSs of both equations are in the range of $\mathbf{A}$. It is obvious that $\mathbf{b} \in range(\mathbf{A})$ for equation 20, since $\mathbf{b} \in range(\mathbf{B})$ and $range(\mathbf{B}) \subset range(\mathbf{A})$. For equation 21, from Proposition 2 we can see that $\mathbf{u}_i$, $1 \leq i \leq p$, is orthogonal to $null(\mathbf{A}^T)$, which means $\mathbf{u}_i \in range(\mathbf{A})$ for $i = 1, \ldots, p$.

(2) As for equation 22, we need to show that $\mathbf{V}^T\bar{\mathbf{x}}$ is in the range of $\mathbf{C}$. We use proof by contradiction in the following. Assume that $\mathbf{V}^T\bar{\mathbf{x}}$ is not in the range of $\mathbf{C}$, then there exists a nonzero vector $\boldsymbol{\gamma} \in null(\mathbf{C}^T)$, i.e. $\mathbf{C}^T\boldsymbol{\gamma} = \mathbf{0}$, such that

$$\boldsymbol{\gamma}^T(\mathbf{V}^T\bar{\mathbf{x}}) \neq 0. \tag{24}$$

Using the generalized inverse [2] $\mathbf{A}^+$ of the matrix $\mathbf{A}$, we can write $\bar{\mathbf{x}}$ as follows.

$$\bar{\mathbf{x}} = \mathbf{A}^+\mathbf{b} + \sum_{i=1}^{m_A} a_i \mathbf{q}'_i, \tag{25}$$

where $m_A = n - rank(\mathbf{A})$, $\mathbf{q}'_1, \ldots, \mathbf{q}'_{m_A}$ are the orthogonal eigenvectors in $null(\mathbf{A})$, and $a_i \in \Re, \forall i$. Then,

$$\mathbf{V}^T\bar{\mathbf{x}} = \mathbf{V}^T\mathbf{A}^+\mathbf{b} + \sum_{i=1}^{m_A} a_i \mathbf{V}^T\mathbf{q}'_i, \tag{26}$$

¿From Proposition 2, we have $\mathbf{V}^T\mathbf{q}'_i = \mathbf{0}$, for $i = 1, \ldots, m_A$. Substituting into equation 26, we have $\mathbf{V}^T\bar{\mathbf{x}} = \mathbf{V}^T\mathbf{A}^+\mathbf{b}$. Substituting for $\mathbf{V}^T\bar{\mathbf{x}}$ in the inequality 24 yields

$$(\mathbf{A}^{+T}\mathbf{V}\boldsymbol{\gamma})^T\mathbf{b} \neq 0. \tag{27}$$

We can show that $\mathbf{A}^{+T}\mathbf{V}\boldsymbol{\gamma} \in null(\mathbf{B}^T)$ as follows.

$$\begin{aligned} \mathbf{B}^T(\mathbf{A}^{+T}\mathbf{V}\boldsymbol{\gamma}) &= (\mathbf{A} + \mathbf{U}\mathbf{G}\mathbf{V}^T)^T\mathbf{A}^{+T}\mathbf{V}\boldsymbol{\gamma} & (28)\\ &= (\mathbf{A}^T\mathbf{A}^{+T}\mathbf{V} + \mathbf{V}\mathbf{G}^T\mathbf{U}^T\mathbf{A}^{+T}\mathbf{V})\boldsymbol{\gamma} & (29) \end{aligned}$$

Since $\mathbf{A}^T\mathbf{A}^{+T} = I - \sum_{i=1}^{m_A} \mathbf{q}'_i\mathbf{q}'^T_i$, using a result of Proposition 2 namely, $\mathbf{q}'^T_i\mathbf{V} = \mathbf{0}$ for $i = 1, \ldots, m_A$, we have

$$\mathbf{A}^T\mathbf{A}^{+T}\mathbf{V} = \mathbf{V} - \sum_{i=1}^{m_A} \mathbf{q}'_i\mathbf{q}'^T_i\mathbf{V} = \mathbf{V}. \tag{30}$$

Substituting equation equation 30 into equation 29 yields

$$\mathbf{B}^T(\mathbf{A}^{+T}\mathbf{V}\boldsymbol{\gamma}) = \mathbf{V}\mathbf{G}^T(\mathbf{G}^{-1} + \mathbf{V}^T\mathbf{A}^+\mathbf{U})^T\boldsymbol{\gamma} \tag{31}$$

Again, using Proposition 2, we can show that $\mathbf{C} = \mathbf{G}^{-1} + \mathbf{V}^T\mathbf{A}^+\mathbf{U}$. Consequently, we obtain $\mathbf{B}^T(\mathbf{A}^{+T}\mathbf{V}\boldsymbol{\gamma}) = \mathbf{V}\mathbf{G}^T\mathbf{C}^T\boldsymbol{\gamma} = \mathbf{0}$ since $\mathbf{C}^T\boldsymbol{\gamma} = \mathbf{0}$ by assumption. Thus, $\mathbf{A}^{+T}\mathbf{V}\boldsymbol{\gamma} \in null(\mathbf{B}^T)$. Combining this with the inequality 27 leads to the conclusion that $\mathbf{b}$ is not in $range(\mathbf{B})$. This is a contradiction to the assumption $\mathbf{b} \in range(\mathbf{B})$. Therefore, the assumption that $\mathbf{V}^T\bar{\mathbf{x}}$ is not in the range of $\mathbf{C}$ is not true. Hence, we can conclude that $\mathbf{V}^T\bar{\mathbf{x}} \in \mathbf{C}$. This proves that there exists a solution to equation 22.

(3) The proof for $\mathbf{B}(\bar{\mathbf{x}} - \boldsymbol{\eta\beta}) = \mathbf{b}$ is very similar to part (3) of the proof for Theorem 1 and is omitted here. ■

[2] Let the singular value decomposition of $\mathbf{A}$ be $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$, where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices and $\boldsymbol{\Sigma} = diag(0, \ldots, 0, \lambda_{m_A+1}, \ldots, \lambda_n)$. Then its generalized inverse $\mathbf{A}^+ = \mathbf{V}\boldsymbol{\Sigma}^+\mathbf{U}^T$, where $\boldsymbol{\Sigma}^+ = diag(0, \ldots, 0, \frac{1}{\lambda_{m_A+1}}, \ldots, \frac{1}{\lambda_n})$ [4].

If we restrict the above constraints on the relation between $\mathbf{B}$ and $\mathbf{A}$ to $null(\mathbf{B}) = null(\mathbf{A})$ and $range(\mathbf{B}) = range(\mathbf{A})$, then the capacitance matrix $\mathbf{C}$ is nonsingular, i.e., the capacitance matrix equation has a unique solution. This result is stated in the following corollary.

Corollary 1 *In Theorem 2, if $null(\mathbf{A}) = null(\mathbf{B})$ and $range(\mathbf{A}) = range(\mathbf{B})$, then the capacitance matrix $\mathbf{C}$ is nonsingular.*

Proof: To prove that $\mathbf{C}$ is nonsingular, we show that $\mathbf{C}\boldsymbol{\beta} = \mathbf{0} \Rightarrow \boldsymbol{\beta} = \mathbf{0}$. Suppose $\mathbf{C}\boldsymbol{\beta} = \mathbf{0}$, then, from the definition of $\mathbf{C}$, we have $\mathbf{V}^T\boldsymbol{\eta}\boldsymbol{\beta} = -\mathbf{G}^{-1}\boldsymbol{\beta}$, which can be substituted into the equation $\mathbf{B}\boldsymbol{\eta}\boldsymbol{\beta} = \mathbf{A}\boldsymbol{\eta}\boldsymbol{\beta} + \mathbf{U}\mathbf{V}^T\boldsymbol{\eta}\boldsymbol{\beta}$, along with the equation for $\mathbf{A}\boldsymbol{\eta}$ from (21), to get $\mathbf{B}\boldsymbol{\eta}\boldsymbol{\beta} = \mathbf{0}$, i.e., $\boldsymbol{\eta}\boldsymbol{\beta} \in null(\mathbf{B})$. Since $null(\mathbf{A}) = null(\mathbf{B})$, we have $\mathbf{A}\boldsymbol{\eta}\boldsymbol{\beta} = \mathbf{0}$. Using equation 21 for $\mathbf{A}\boldsymbol{\eta}$, we get $\mathbf{U}\boldsymbol{\beta} = \mathbf{0}$. ¿From the assumption on the matrix $\mathbf{U}$ namely, $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are linearly independent, we conclude that $\boldsymbol{\beta} = \mathbf{0}$. Therefore, $\mathbf{C}$ is nonsingular. ■

In the above theorems, we impose the constraints on the relationship between $\mathbf{B}$ and $\mathbf{A}$ either via $null(\mathbf{B}) \subset null(\mathbf{A})$ and $range(\mathbf{A}) \subset range(\mathbf{B})$ or via $null(\mathbf{A}) \subset null(\mathbf{B})$ and $range(\mathbf{B}) \subset range(\mathbf{A})$. Now we consider the general case, i.e., no constraint on the relationship between $\mathbf{B}$ and $\mathbf{A}$ is imposed. The following theorem is given for this general case and can be used for any matrices $\mathbf{B}$ and $\mathbf{A}$ in $\Re^{n\times n}$ as long as there exists a solution for the associated capacitance matrix equation.

Theorem 3 *Given $\mathbf{B} = \mathbf{A} + \mathbf{U}\mathbf{G}\mathbf{V}^T$, where $\mathbf{B}, \mathbf{A} \in \Re^{n\times n}$, $\mathbf{G} \in \Re^{p\times p}$ is nonsingular, $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_p] \in \Re^{n\times p}$, $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_p] \in \Re^{n\times p}$, with both the sets $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ containing linearly independent vectors. Let $rank(\mathbf{A}) = n - m_A$, $dim(null(\mathbf{B}) \cap null(\mathbf{A})) = m'$, and $dim(null(\mathbf{B}^T) \cap null(\mathbf{A}^T)) = m$, $0 \le m', m \le m_A \le n$, then there exist two sets of orthogonal basis, $\{\mathbf{q}_1, \ldots, \mathbf{q}_{m_A}\}$ and $\{\mathbf{q}'_1, \ldots, \mathbf{q}'_{m_A}\}$, for $null(\mathbf{A}^T)$ and $null(\mathbf{A})$ respectively, such that $null(\mathbf{B}^T) \cap null(\mathbf{A}^T) = span\{\mathbf{q}_{m_A-m+1}, \ldots, \mathbf{q}_{m_A}\}$ and $null(\mathbf{B}) \cap null(\mathbf{A}) = span\{\mathbf{q}'_{m_A-m'+1}, \ldots, \mathbf{q}'_{m_A}\}$. Let $\mathbf{q}_i^T\mathbf{u}_j = \alpha_i\delta(i,j)$, $\alpha_i \neq 0$, for $1 \le i, j \le m_A - m$.*

Let $\bar{\mathbf{x}}$ be a solution to

$$\mathbf{A}\bar{\mathbf{x}} = \mathbf{b} - \sum_{j=1}^{m_A-m} \frac{\mathbf{q}_j^T\mathbf{b}}{\mathbf{q}_j^T\mathbf{u}_j}\mathbf{u}_j, \tag{32}$$

where $\mathbf{b} \in range(\mathbf{B})$. For $1 \le i \le p$, let $\boldsymbol{\eta}_i$ be a solution to

$$\mathbf{A}\boldsymbol{\eta}_i = \mathbf{u}_i - \sum_{j=1}^{m_A-m} \frac{\mathbf{q}_j^T\mathbf{u}_i}{\mathbf{q}_j^T\mathbf{u}_j}\mathbf{u}_j. \tag{33}$$

Define the capacitance matrix $\mathbf{C}$ as

$$\mathbf{C} = \mathbf{G}^{-1} + \mathbf{V}^{\mathbf{T}}\boldsymbol{\eta} - \sum_{j=1}^{m_A-m} \mathbf{G}^{-1}\mathbf{e}_j\frac{\mathbf{q}_j^T\mathbf{U}}{\mathbf{q}_j^T\mathbf{u}_j}, \tag{34}$$

where $\boldsymbol{\eta} = [\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_p]$, and $\mathbf{e}_j$ is the j-th standard basis vector in $\Re^p$. Assume there exists a solution $\boldsymbol{\beta}$ to the capacitance matrix equation

$$\mathbf{C}\boldsymbol{\beta} = \mathbf{V}^T\bar{\mathbf{x}} - \sum_{j=1}^{m_A-m} \frac{\mathbf{q}_j^T\mathbf{b}}{\mathbf{q}_j^T\mathbf{u}_j}\mathbf{G}^{-1}\mathbf{e}_j, \tag{35}$$

then $\bar{\mathbf{x}} - \boldsymbol{\eta\beta}$ *is a solution to the linear system* $\mathbf{Bx} = \mathbf{b}$.

The proof of this theorem is omitted here since, it is similar to the proof of Theorem 1. This proof will require the use of the fact that $\mathbf{q}_i^T\mathbf{u}_j = 0$, for $m_A - m + 1 \leq i \leq m_A$ and $1 \leq j \leq p$.

In general, Theorems 1 and 2 can be regarded as the special cases for Theorem 3 with the exception of the existence of a solution to the capacitance matrix equation. In the first two theorems, there always exists a solution for the capacitance matrix equation, while the existence of such solution is assumed in Theorem 3. We cannot guarantee the capacitance matrix equation always has a solution for the general case; but, with certain constraints, we can prove it does have a solution. This is given in the following corollary.

Corollary 2 *In Theorem 3, assume* $\mathbf{q}'_j \in range(\mathbf{V})$ *for* $1 \leq j \leq m_A - m'$. *Let* $\bar{\mathbf{x}}^T\mathbf{q}'_j = a_j$ *and* $\boldsymbol{\eta}_i^T\mathbf{q}'_j = d_{ij}$, $1 \leq i \leq p$ *and* $1 \leq j \leq m_A - m'$. *If the coefficients* a_j *and* d_{ij} *are chosen such that* $a_j = 0$ *when* $\mathbf{q}'^T_j\mathbf{b} = 0$ *and the equation*

$$\mathbf{VG}^T\mathbf{d}_j = a'_j\mathbf{VG}^T\mathbf{U}^T\mathbf{q}'_j - \mathbf{q}'_j \tag{36}$$

is satisfied for $j = 1, \ldots, m_A - m'$, *where* $\mathbf{d}_j = (d_{1j} \ldots d_{pj})^T$ *and*

$$a'_j := \begin{cases} \frac{a_j}{\mathbf{q}'^T_j\mathbf{b}} & when \;\; \mathbf{q}'^T_j\mathbf{b} \neq 0 \\ 0 & elsewhere \end{cases},$$

then, there always exists a solution to equation 35.

Proof: To prove the existence of a solution for equation 35, we need to show that the RHS of equation 35 is in the range of $\mathbf{C}$ for which we will use proof by contradiction. Suppose the RHS of equation 35 is not in the range of $\mathbf{C}$, then there exists a nonzero vector $\boldsymbol{\gamma} \in null(\mathbf{C}^T)$, i.e. $\mathbf{C}^T\boldsymbol{\gamma} = \mathbf{0}$, such that

$$\boldsymbol{\gamma}^T(\mathbf{V}^T\bar{\mathbf{x}} - \sum_{j=1}^{m_A-m} \frac{\mathbf{q}_j^T\mathbf{b}}{\mathbf{q}_j^T\mathbf{u}_j}\mathbf{G}^{-1}\mathbf{e}_j) \neq 0. \tag{37}$$

Denote the generalized inverse of the matrix $\mathbf{A}$ by $\mathbf{A}^+$, then we have

$$\mathbf{V}^T\bar{\mathbf{x}} = \mathbf{V}^T\mathbf{A}^+\mathbf{b} - \mathbf{V}^T\mathbf{A}^+ \sum_{j=1}^{m_A-m} \frac{\mathbf{q}_j^T\mathbf{b}}{\mathbf{q}_j^T\mathbf{u}_j}\mathbf{u}_j + \sum_{j=1}^{m_A-m'} a_j\mathbf{V}^T\mathbf{q}'_j. \tag{38}$$

Substituting equation 38 into inequality 37 and using the definition of a'_j, the inequality can be rearranged as follows

$$(\mathbf{A}^{+T}\mathbf{V}\boldsymbol{\gamma} - \sum_{j=1}^{m_A-m} \frac{\mathbf{q}_j\mathbf{u}_j^T\mathbf{A}^{+T}\mathbf{V}}{\mathbf{q}_j^T\mathbf{u}_j}\boldsymbol{\gamma} + \sum_{j=1}^{m_A-m'} a'_j\mathbf{q}'_j\mathbf{q}'^T_j\mathbf{V}\boldsymbol{\gamma} - \sum_{j=1}^{m_A-m} \frac{\mathbf{q}_j\mathbf{e}_j^T\mathbf{G}^{-T}}{\mathbf{q}_j^T\mathbf{u}_j}\boldsymbol{\gamma})^T\mathbf{b} \neq 0. \tag{39}$$

Denote the vector in the parenthesis by $\boldsymbol{\xi}$ i.e.,

$$\boldsymbol{\xi} := \mathbf{A}^{+T}\mathbf{V}\boldsymbol{\gamma} - \sum_{j=1}^{m_A-m} \frac{\mathbf{q}_j\mathbf{u}_j^T\mathbf{A}^{+T}\mathbf{V}}{\mathbf{q}_j^T\mathbf{u}_j}\boldsymbol{\gamma} + \sum_{j=1}^{m_A-m'} a'_j\mathbf{q}'_j\mathbf{q}'^T_j\mathbf{V}\boldsymbol{\gamma} - \sum_{j=1}^{m_A-m} \frac{\mathbf{q}_j\mathbf{e}_j^T\mathbf{G}^{-T}}{\mathbf{q}_j^T\mathbf{u}_j}\boldsymbol{\gamma}.$$

In the following, we show that $\boldsymbol{\xi} \in null(\mathbf{B}^T)$. Simplifying the expansion of $\mathbf{B}^T\boldsymbol{\xi}$ yields

$$\begin{aligned}\mathbf{B}^T\boldsymbol{\xi} &= (\mathbf{A}^T\mathbf{A}^{+T}\mathbf{V} + \mathbf{V}\mathbf{G}^T\mathbf{U}^T\mathbf{A}^{+T}\mathbf{V} - \mathbf{V}\mathbf{G}^T \sum_{j=1}^{m_A-m} \frac{\mathbf{U}^T\mathbf{q}_j\mathbf{u}_j^T\mathbf{A}^{+T}\mathbf{V}}{\mathbf{q}_j^T\mathbf{u}_j} + \\ &\quad \sum_{j=1}^{m_A-m'} a'_j\mathbf{V}\mathbf{G}^T\mathbf{U}^T\mathbf{q}'_j\mathbf{q}'^T_j\mathbf{V} - \mathbf{V}\mathbf{G}^T \sum_{j=1}^{m_A-m} \frac{\mathbf{U}^T\mathbf{q}_j\mathbf{e}_j^T\mathbf{G}^{-T}}{\mathbf{q}_j^T\mathbf{u}_j})\boldsymbol{\gamma}\end{aligned}$$

By substituting the identity $\mathbf{A}^T\mathbf{A}^{+T}\mathbf{V} = \mathbf{V} - \sum_{j=1}^{m_A-m'} \mathbf{q}'_j\mathbf{q}'^T_j\mathbf{V}$ and the assumption of equation 36 into the above equation, we get

$$\mathbf{B}^T\boldsymbol{\xi} = \mathbf{V}\mathbf{G}^T(\mathbf{G}^{-1}+\mathbf{V}^T\mathbf{A}^+\mathbf{U}-\mathbf{V}^T\mathbf{A}^+ \sum_{j=1}^{m_A-m} \mathbf{u}_j\frac{\mathbf{q}_j^T\mathbf{U}}{\mathbf{q}_j^T\mathbf{u}_j}+ \sum_{j=1}^{m_A-m'} \mathbf{V}^T\mathbf{q}'_j\mathbf{d}_j^T - \sum_{j=1}^{m_A-m} \mathbf{G}^{-1}\mathbf{e}_j\frac{\mathbf{q}_j^T\mathbf{U}}{\mathbf{q}_j^T\mathbf{u}_j})^T\boldsymbol{\gamma}. \quad (40)$$

Substituting $\mathbf{V}^T\boldsymbol{\eta}_i$ by $\mathbf{V}^T\mathbf{A}^+\mathbf{u}_i - \sum_{j=1}^{m_A-m} \frac{\mathbf{q}_j^T\mathbf{u}_i}{\mathbf{q}_j^T\mathbf{u}_j}\mathbf{V}^T\mathbf{A}^+\mathbf{u}_j + \sum_{j=1}^{m_A-m'} d_{ij}\mathbf{V}^T\mathbf{q}'_j$, we can see that the matrix in parenthesis of equation 40 is equivalent to the capacitance matrix $\mathbf{C}$. ¿From the assumption $\mathbf{C}^T\boldsymbol{\gamma} = \mathbf{0}$, we have $\mathbf{B}^T\boldsymbol{\xi} = \mathbf{0}$, i.e., $\boldsymbol{\xi} \in null(\mathbf{B}^T)$. Combining this with the inequality 39 leads to the result that $\mathbf{b}$ is not in $range(\mathbf{B})$. This contradicts the assumption $\mathbf{b} \in range(\mathbf{B})$. Therefore, the assumption in the beginning of the proof that $\mathbf{V}^T\bar{\mathbf{x}}$ is not in the range of $\mathbf{C}$ is not true. Thus we can conclude that $\mathbf{V}^T\bar{\mathbf{x}} \in \mathbf{C}$. This proves that there exists a solution to equation 35. ■

3 Discussion and Summary

From the theorems presented in section 2, we can modify the capacitance matrix method as follows.

1. Solve equation 32 for $\bar{\mathbf{x}}$.
2. Solve equation 33 for $\boldsymbol{\eta}_i$, for $i = 1, \ldots, p$.
3. Form the capacitance matrix $\mathbf{C}$ from equation 34.
4. Solve the capacitance matrix equation (i.e. eq. 35) for $\boldsymbol{\beta}$.
5. The solution $\mathbf{x} = \bar{\mathbf{x}} - \boldsymbol{\eta}\boldsymbol{\beta}$.

This modified capacitance matrix method can always be applied to the cases when $null(\mathbf{B}) \subset null(\mathbf{A})$ and $range(\mathbf{A}) \subset range(\mathbf{B})$ or when $null(\mathbf{A}) \subset null(\mathbf{B})$ and $range(\mathbf{B}) \subset range(\mathbf{A})$, as

stated in Theorem 1 and 2 respectively. However, to apply this method to more general cases, we need to make sure that a solution exists to the capacitance matrix equation in step 4. Corollary 2 gives a sufficient condition for the existence of the solution to the capacitance matrix equation under the general case, but no necessary condition for the existence of a solution to the capacitance matrix equation is available at this time.

We have successfully applied this *modified capacitance matrix method* to a host of low-level vision problems which use the membrane or thin-plate spline to constrain the solution [14, 7]. These low-level vision problems include the surface reconstruction, shape from orientation, optical flow, lightness problem, etc.[14, 7]. These problems when formulated as variational principles and discretized, lead to solving one or more large sparse linear systems. For the membrane spline, we can choose the well-structured matrix $\mathbf{A}$, which is close to the matrix $\mathbf{B}$, i.e. the stiffness matrix in the large linear system, to be the Laplacian matrix, thus making the matrix $\mathbf{UV}^T$ encode the components of the stiffness matrix corresponding to the data constraints, discontinuities, and boundary conditions . The resulting matrices $\mathbf{A}$ and $\mathbf{B}$ either satisfy the condition $null(\mathbf{B}) \subset null(\mathbf{A})$ and $range(\mathbf{A}) \subset range(\mathbf{B})$ in Theorem 1 (for surface reconstruction and optical flow) or satisfy the condition $null(\mathbf{A}) \subset null(\mathbf{B})$ and $range(\mathbf{B}) \subset range(\mathbf{A})$ in Theorem 2 (for shape from orientation and lightness problem).

The traditional capacitance matrix method [2, 9, 12, 13, 8, 10, 3, 1] has been primarily applied to solve the second-order elliptic boundary value problems, whereas our modified capacitance matrix method (presented above) can be used to solve higher-order elliptic boundary value problems. Furthermore, efficient numerical algorithms can be designed [14, 7] based on an appropriate choice of the matrix $\mathbf{A}$.

Some works on the extension of the Sherman-Morrison-Woodbury formula in equation 1 to singular $\mathbf{A}$, $\mathbf{B}$, and the associated capacitance matrix $\mathbf{C}$ have been reported in [6], [11], and references therein. However, the generalized inverse forms of $(\mathbf{A} + \mathbf{UGV}^T)$ have been derived under *restrictive assumptions*. Although we can make use of these generalized inverse formulas to design the corresponding capacitance matrix methods, these *assumptions* restrict them from being applied to general linear systems. Our generalized capacitance matrix theorems are developed for solving general linear systems, and it is possible to derive the generalized inverse form of $(\mathbf{A} + \mathbf{UGV}^T)$ for more general cases based on these theorems, which will be the focus of our future research.

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